

A Common Framework for Attitude Synchronization of Unit Vectors in Networks with Switching Topology

Pedro O. Pereira, Dimitris Boskos and Dimos V. Dimarogonas

Abstract—In this paper, we study attitude synchronization for elements in the unit sphere of \mathbb{R}^3 and for elements in the 3D rotation group, for a network with switching topology. The agents' angular velocities are assumed to be the control inputs, and a switching control law for each agent is devised that guarantees synchronization, provided that all elements are initially contained in a given region, unknown to the network. The control law is decentralized and it does not require a common orientation frame among all agents. We refer to synchronization of unit vectors in \mathbb{R}^3 as *incomplete synchronization*, and of 3D rotation matrices as *complete synchronization*. Our main contribution lies in showing that these two problems can be analyzed under a common framework, where all elements' dynamics are transformed into unit vectors dynamics on a sphere of appropriate dimension.

I. INTRODUCTION

Decentralized control in a multi-agent environment has been a topic of active research, with applications in large scale robotic systems. Attitude synchronization in satellite formations is one of the relevant applications [1], [2], where the control goal is to guarantee that a network of fully actuated rigid bodies acquires a common attitude. Coordination of underwater vehicles in ocean exploration missions [3], and of unmanned aerial vehicles in aerial exploration missions, may also be casted as attitude synchronization problems.

In the literature, attitude synchronization strategies for elements in the special orthogonal group are found in [4]–[12], which focus on *complete* attitude synchronization; and in [13]–[23], which focus on *incomplete* attitude synchronization. In this paper, we focus on both *complete* and *incomplete* attitude synchronization. We refer to *incomplete* attitude synchronization when the agents are unit vectors in \mathbb{R}^3 , a space also called \mathbb{S}^2 ; and we refer to *complete* attitude synchronization, when the agents are 3D rotation matrices, a space also called $\mathbb{SO}(3)$. Incomplete synchronization represents a relevant practical problem, when the goal among multiple agents is to share a common direction. In flocking, for example, moving along a common direction is a requirement. Also, in a network of satellites, whose antennas are to point in a common direction, incomplete synchronization may be more important than complete.

In [5]–[8], [10]–[12], state dependent control laws for torques are presented which guarantee synchronization for elements in $\mathbb{SO}(3)$, while in [16], [17], [24] state dependent

control laws for torques are presented which guarantee synchronization for elements in \mathbb{S}^2 . In these works, all agents are dynamic and their angular velocities are part of the state of the system, rather than control inputs. In this paper, however, we consider kinematic agents and we design control laws, for the agents' angular velocities, which are not exclusively state dependent, but are also time dependent, with the time dependency encapsulating the case of a switching network topology.

We note that, regarding synchronization in $\mathbb{SO}(3)$, relevant results are found in [20], [25]–[28]. In [20], [27]–[29], consensus on non-linear spaces is analyzed with the help of a common weak non-smooth Lyapunov function, i.e., a Lyapunov function which is non-increasing along solutions. Also, in [25], control laws which guarantee synchronization under a switching topology are presented, under the hypothesis of a dwell time between consecutive switches.

In our proposed framework, we relax the assumption of a dwell time by providing conditions for synchronization under average dwell time. Our approach is based on the construction of a common weak non-smooth Lyapunov function for analyzing synchronization in \mathbb{S}^n . In order to handle the non-smoothness of the proposed Lyapunov function, we present an invariance-like result (see also [30]–[33] for invariance like theorems for switched systems). We propose control laws for angular velocities of unit vectors in \mathbb{R}^3 and 3D rotation matrices that guarantee synchronization for a network of agents with switching topology. The control laws devised for unit vectors and rotation matrices achieve different goals, and differ in two aspects worth emphasizing. First, controlling rotation matrices requires more measurements when compared with controlling unit vectors; secondly, while controlling rotation matrices requires full actuation, i.e., all body components of the angular velocity need to be controllable, controlling unit vectors does not. Our main contribution compared to the aforementioned literature lies in analyzing both problems under a common framework, in order to allow for a unified stability analysis using the same common weak Lyapunov function. Particularly both problems are transformed into synchronization problems in \mathbb{S}^m for an appropriate $m \in \mathbb{N}$. Since rotation matrices can be parametrized by unit quaternions [34], which are unit vectors in \mathbb{R}^4 , these are chosen for the analysis of the proposed control law. We also note that consensus in \mathbb{R}^n can be casted as a synchronization problem in \mathbb{S}^n . We note that under our framework, we do not require a dwell time between consecutive switches. A preliminary version of this work was presented at the 2016 IEEE Conference on

The authors are with the School of Electrical Engineering, KTH Royal Institute of Technology, SE-100 44, Stockholm, Sweden. Email addresses: {ppereira, boskos, dimos}@kth.se. This work was supported by the EU H2020 Research and Innovation Programme under GA No.644128 (AEROWORKS), the Swedish Research Council (VR), the Swedish Foundation for Strategic Research (SSF) and the KAW Foundation.

Decision and Control [35]. With respect to this preliminary version, this paper provides significant additional results. In particular, more details on the derivation of the main theorems and propositions are presented; additional figures illustrating concepts and results are provided; and supplementary simulations are presented which further illustrate the theoretical results.

The remainder of this paper is structured as follows. In Sections II and III, we present the notation used in this manuscript and conditions on vector fields that guarantee convergence to the consensus set, respectively. In Section IV, we describe the common framework for analysis of both synchronization in \mathbb{S}^2 and $\mathbb{SO}(3)$. In Sections IV-B and IV-A, the control laws for synchronization in $\mathbb{SO}(3)$ and \mathbb{S}^2 are presented, respectively, and the agents dynamics are transformed into the common framework. In Section V, asymptotic synchronization is established for the common framework vector field. In Section VI, illustrative simulations are presented.

II. NOTATION

In what follows, let n and m be two integers. $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}$ denotes the inner product in \mathbb{R}^n . $\mathbb{R}^{m \times n}$ denotes the set of linear mappings from \mathbb{R}^n to \mathbb{R}^m , and with $\mathbf{I}_n \in \mathbb{R}^{n \times n}$ as the identity matrix and $\mathbf{0}_{m \times n} \in \mathbb{R}^{m \times n}$ as the zero matrix. We denote by $\bar{M}_{3,3}$ and $\bar{M}_{3,3}$ the space of symmetric matrices and antisymmetric matrices in $\mathbb{R}^{3 \times 3}$, respectively. Given $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$, the matrix $\mathcal{S}(\mathbf{a}) \in \bar{M}_{3,3}$ is the skew-symmetric matrix that satisfies $\mathcal{S}(\mathbf{a}) \mathbf{b} = \mathbf{a} \times \mathbf{b}$; moreover, given any $A \in \bar{M}_{3,3}$, $\mathcal{S}^{-1}(A) \in \mathbb{R}^3$ denotes the vector for which $\mathcal{S}(\mathcal{S}^{-1}(A)) = A$. We denote by $\mathbb{S}^n = \{\mathbf{x} \in \mathbb{R}^{n+1} : \langle \mathbf{x}, \mathbf{x} \rangle = 1\}$ the set of unit vectors in \mathbb{R}^{n+1} . The map $\Pi : \mathbb{S}^n \ni \mathbf{x} \mapsto \Pi(\mathbf{x}) \in \mathbb{R}^{(n+1) \times (n+1)}$, yields a matrix that satisfies $\Pi(\mathbf{x}) \mathbf{y} = \mathbf{y} - \langle \mathbf{y}, \mathbf{x} \rangle \mathbf{x}$ for any $\mathbf{y} \in \mathbb{R}^3$, and it represents the orthogonal projection operator onto the subspace orthogonal to $\mathbf{x} \in \mathbb{S}^n$. $\mathbf{e}_1, \dots, \mathbf{e}_n \in \mathbb{S}^{n-1} \subset \mathbb{R}^n$ denote the canonical basis vectors in \mathbb{R}^n . Given $r \geq 0$, we denote $\mathcal{B}(r) = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\| < r\}$ and $\bar{\mathcal{B}}(r) = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\| \leq r\}$ as the open and closed balls of radius r and centered around $\mathbf{0}$, respectively. Given a set \mathcal{H} , we use the notation $|\mathcal{H}|$ for the cardinality of \mathcal{H} . Given a function $f : A \ni a \mapsto f(a) \in B$, for some arbitrary domain A and codomain B : if f is differentiable df denotes its derivative; and given any $N \in \mathbb{N}$, we denote $f^N : A^N \ni a := (a_1, \dots, a_N) \mapsto f^N(a) := (f(a_1), \dots, f(a_N)) \in B^N$. Finally, given a manifold M , $T_{\mathbf{m}}M$ denotes the tangent space to M at $\mathbf{m} \in M$.

III. PRELIMINARIES

We consider a network of $N \in \mathbb{N}$ agents, with $n \in \mathbb{N}$ the dimension of the space which the agents belong to. We associate bijectively the (finite) set of network graphs to the set $\{1, 2, \dots\} =: \mathcal{P} \subset \mathbb{N}$, and consider an average dwell time switching signal $\sigma : \mathbb{R}_{\geq 0} \mapsto \mathcal{P}$ [32]. We then consider a trajectory

$$\mathbb{R}_{\geq 0} \ni t \mapsto \mathbf{x}(t) := (\mathbf{x}_1(t), \dots, \mathbf{x}_N(t)) \in (\mathbb{R}^n)^N \quad (1)$$

which satisfies

$$\dot{\mathbf{x}}(t) = \mathbf{f}_{\sigma(t)}(\mathbf{x}(t)), \forall t \in \mathbb{R}_{\geq 0}, \quad (2)$$

and where, for any $p \in \mathcal{P}$,

$$\mathbf{f}_p := (\mathbf{f}_{1,p}, \dots, \mathbf{f}_{N,p}) : (\mathbb{R}^n)^N \mapsto (\mathbb{R}^n)^N. \quad (3)$$

We now present conditions on the vector fields $\{\mathbf{f}_p\}_{p \in \mathcal{P}}$ in (3) which guarantee that \mathbf{x} in (1) converges to the consensus set $\mathcal{C} = \{(\mathbf{x}_1, \dots, \mathbf{x}_N) \in (\mathbb{R}^n)^N : \mathbf{x}_1 = \dots = \mathbf{x}_N\}$. In later sections, when studying synchronization in \mathbb{S}^2 and $\mathbb{SO}(3)$, given the proposed control laws, we verify that the dynamics of all agents satisfy these conditions, allowing us to refer to the results in this section.

Given $\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_N) \in (\mathbb{R}^n)^N$, denote

$$\mathcal{H}(\mathbf{x}) := \arg \max_{j \in \mathcal{N}} (\|\mathbf{x}_j\|) \subseteq \mathcal{N}. \quad (4)$$

Then, given $\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_N) \in (\mathbb{R}^n)^N$ and $i \in \mathcal{H}(\mathbf{x})$, $\|\mathbf{x}_i\|$ is larger than or equal to $\|\mathbf{x}_j\|$ for all $j \in \mathcal{N}$. Given $\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_N) \in (\mathbb{R}^n)^N$ assume that

$$\max_{p \in \mathcal{P}} \langle \mathbf{x}_i, \mathbf{f}_{i,p}(\mathbf{x}) \rangle \leq 0, \text{ for all } i \in \mathcal{H}(\mathbf{x}), \quad (5)$$

which quantifies an upper bound on the time derivative of $\frac{1}{2}\|\mathbf{x}_i\|^2$ along a solution that satisfies (2), since $\frac{d}{dt} \frac{1}{2}\|\mathbf{x}_i(t)\|^2 = \langle \mathbf{x}_i(t), \mathbf{f}_{i,\sigma(t)}(\mathbf{x}(t)) \rangle$, for all time instants t where the derivative is well defined. On the other hand, given $\mathbf{x} \notin \mathcal{C}$, assume that (5) holds and that

$$\forall p \in \mathcal{P} \exists i \in \mathcal{H}(\mathbf{x}) : \langle \mathbf{x}_i, \mathbf{f}_{i,p}(\mathbf{x}) \rangle < 0. \quad (6)$$

Condition (6) implies that, for every switching signal $p \in \mathcal{P}$, and within the set $\{\frac{1}{2}\|\mathbf{x}_i\|^2\}_{i \in \mathcal{H}(\mathbf{x})}$, one can always find an element whose time derivative along a solution of (2), if well defined, is negative. Conditions (5) and (6) are central for the proof of the following theorem, which establishes asymptotic convergence of (1) to \mathcal{C} and constitutes the main result of this section.

Theorem 1: Consider (1) and assume that for certain $r > 0$ and all $\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_N) \in \mathcal{B}(r)^N$ the following hold:

- 1) when $\mathbf{x} \notin \mathcal{C}$,
 - a) $\max_{p \in \mathcal{P}} \langle \mathbf{x}_i, \mathbf{f}_{i,p}(\mathbf{x}) \rangle \leq 0$, for all $i \in \mathcal{H}(\mathbf{x})$,
 - b) $\forall p \in \mathcal{P} \exists i \in \mathcal{H}(\mathbf{x}) : \langle \mathbf{x}_i, \mathbf{f}_{i,p}(\mathbf{x}) \rangle < 0$,
- 2) when $\mathbf{x} \in \mathcal{C}$, $\langle \mathbf{x}_i, \mathbf{f}_{i,p}(\mathbf{x}) \rangle = 0$ for all $i \in \mathcal{N}$ and $p \in \mathcal{P}$.

Then, the set $\bar{\mathcal{B}}(r_0)^N$, with $r_0 := \max_{i \in \mathcal{N}} \|\mathbf{x}_i(0)\| < r$, is positively invariant. Moreover, for each initial condition $\mathbf{x}(0) \in \mathcal{B}(r)^N$, and given $V : (\mathbb{R}^n)^N \ni \mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_N) \mapsto V(\mathbf{x}) := \max_{i \in \mathcal{N}} \frac{1}{2}\|\mathbf{x}_i\|^2 \in \mathbb{R}_{\geq 0}$, there exists a constant $V^\infty \in [0, V(\mathbf{x}(0))]$ such that $\lim_{t \rightarrow \infty} V(\mathbf{x}(t)) = V^\infty$. Finally, (1) converges asymptotically to $V^{-1}(V^\infty) \cap \mathcal{C} \subset \bar{\mathcal{B}}(r_0)^N \cap \mathcal{C}$.

The proof is found in Appendix I.

IV. SYNCHRONIZATION

In the next subsections, we study synchronization of agents in \mathbb{S}^2 and $\mathbb{SO}(3)$. More specifically, we first present feedback control laws for the angular velocities of the agents, with which we determine the closed loop dynamics. Afterwards,

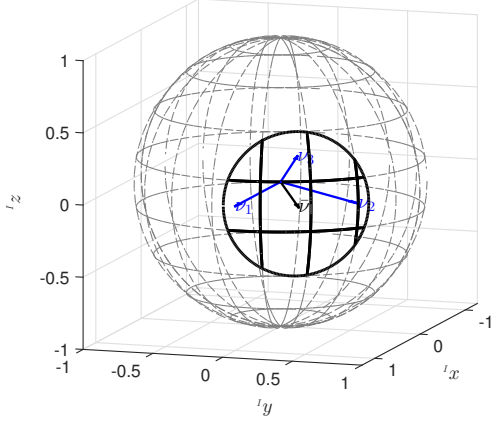


Fig. 1: Three unit vectors, ν_1 , ν_2 and ν_3 , in \mathbb{R}^3 contained in closed 30° -cone formed by the unit vector $\bar{\nu}$.

by means of appropriate transformations, those dynamics are rewritten in a common form that allows us to study synchronization in \mathbb{S}^2 and $\mathbb{SO}(3)$ under a common framework. Additionally, in Appendix V, we also show that consensus in \mathbb{R}^n can be casted as a synchronization problem in \mathbb{S}^n , for any $n \in \mathbb{N}$.

Definition 1: Given $n \in \mathbb{N}$, $\alpha \in [0, \pi]$ and $\bar{\nu} \in \mathbb{S}^n$, the open α -cone $\mathcal{C}(\alpha, \bar{\nu})$ is defined as $\mathcal{C}(\alpha, \bar{\nu}) := \{\nu \in \mathbb{S}^n : \langle \bar{\nu}, \nu \rangle > \cos(\alpha)\}$, representing the set of unit vectors that are α close to the unit vector $\bar{\nu}$. Similarly, we define the closed α -cone $\bar{\mathcal{C}}(\alpha, \bar{\nu}) := \{\nu \in \mathbb{S}^n : \langle \bar{\nu}, \nu \rangle \geq \cos(\alpha)\}$.

In Fig. 1, we illustrate a closed cone, for $n = 3$, where three unit vectors ν_1 , ν_2 and ν_3 are contained in the closed 30° -cone formed by the unit vector $\bar{\nu}$.

Consider a group of unit vectors $\nu = (\nu_1, \dots, \nu_N) \in (\mathbb{S}^n)^N$, for some $N, n \in \mathbb{N}$. We say that ν belongs to an open (closed) α -cone, for some $\alpha \in [0, \pi]$, if $\exists \bar{\nu} \in \mathbb{S}^n : \nu \in \mathcal{C}(\alpha, \bar{\nu})^N (\bar{\mathcal{C}}(\alpha, \bar{\nu})^N)$. We say that ν is synchronized if $\nu_1 = \dots = \nu_N$. We show later that, given the proposed control laws, synchronization of a group of unit vectors takes place asymptotically if the group of unit vectors is initially contained in an open α^* -cone, where $\alpha^* = \frac{\pi}{2}$ for synchronization in \mathbb{S}^2 and consensus in \mathbb{R}^n ; and where $\alpha^* = \frac{\pi}{4}$ for synchronization in $\mathbb{SO}(3)$.

In the next subsections, we always consider a group of N agents, indexed by the set $\mathcal{N} := \{1, \dots, N\}$, operating in either \mathbb{S}^2 , $\mathbb{SO}(3)$ or \mathbb{R}^n . The agents' network graph is modeled as a time varying digraph, $\mathbb{R}_+ \ni t \mapsto \mathcal{G}(\sigma(t)) = \{\mathcal{N}, \mathcal{E}(\sigma(t))\}$, with $\sigma : \mathbb{R}_{\geq 0} \mapsto \mathcal{P}$ as the switching signal, and with $\mathcal{G}(p)$ and $\mathcal{E}(p)$ as the graph and edges' set corresponding to the switching signal $p \in \mathcal{P}$ (where $|\mathcal{P}| \leq 2^{N(N-1)}$). We also denote $\mathcal{N}_i(p) \subset \mathcal{N}$ as the neighbor set of agent $i \in \mathcal{N}$ corresponding to the switching signal $p \in \mathcal{P}$; and, for convenience, we also denote $\{j_1, \dots, j_{|\mathcal{N}_i(q)|}\} \equiv \mathcal{N}_i(q)$.

In order to perform analysis under a common framework, we transform all problems' dynamics into a standard form, which we describe next. Given $n \in \mathbb{N}$, we denote $\nu =$

$(\nu_1, \dots, \nu_N) : \mathbb{R}_{\geq 0} \mapsto (\mathbb{S}^n)^N$ as the state of a group of unit vectors in \mathbb{S}^n , which evolves according to the dynamics

$$\dot{\nu}(t) = \tilde{\mathbf{f}}_{\sigma(t)}(\nu(t)) = \begin{bmatrix} \tilde{\mathbf{f}}_{1,\sigma(t)}(\nu(t)) \\ \vdots \\ \tilde{\mathbf{f}}_{N,\sigma(t)}(\nu(t)) \end{bmatrix}, \nu(0) \in (\mathbb{S}^n)^N \quad (7)$$

where $\tilde{\mathbf{f}}_{i,\sigma(t)} : (\mathbb{S}^n)^N \ni \nu := (\nu_1, \dots, \nu_N) \mapsto \tilde{\mathbf{f}}_{i,\sigma(t)}(\nu) \in T_{\nu_i} \mathbb{S}^n \subset \mathbb{R}^{n+1}$ is defined as

$$\tilde{\mathbf{f}}_{i,\sigma(t)}(\nu) := \sum_{j \in \mathcal{N}_i(\sigma(t))} \tilde{w}_{ij}(\nu_i, \nu_j) \Pi(\nu_i) \nu_j, \quad (8)$$

for all $i \in \mathcal{N}$; i.e., $\dot{\nu}_i(t) = \tilde{\mathbf{f}}_{i,\sigma(t)}(\nu(t))$. Notice that indeed $\langle \nu_i, \tilde{\mathbf{f}}_{i,p}(\nu) \rangle = 0$ for all $i \in \mathcal{N}$, $p \in \mathcal{P}$ and $\nu = (\nu_1, \dots, \nu_N) \in (\mathbb{S}^n)^N$, which implies that the set $(\mathbb{S}^n)^N$ is positively invariant with respect to (7).

The system (7)-(8) is the standard form all problems are transformed into: for synchronization in \mathbb{S}^2 , $\nu : \mathbb{R}_{\geq 0} \mapsto (\mathbb{S}^2)^N$; for synchronization in $\mathbb{SO}(3)$, $\nu : \mathbb{R}_{\geq 0} \mapsto (\mathbb{S}^3)^N$; and for consensus in \mathbb{R}^n , $\nu : \mathbb{R}_{\geq 0} \mapsto (\mathbb{S}^n)^N$.

The functions $\tilde{w}_{ij} : \mathcal{C}(\alpha, \bar{\nu})^2 \mapsto \mathbb{R}_{\geq 0}$ in (8) are continuous weight functions, for some $\alpha \in [\frac{\pi}{2}, \pi]$ and $\bar{\nu} \in \mathbb{S}^n$. Thus, given $(\nu_i, \nu_j) \in \mathcal{C}(\alpha, \bar{\nu})^2$, $\tilde{w}_{ij}(\nu_i, \nu_j)$ is the weight agent i assigns to the deviation between itself and its neighbor j , for all $i, j \in \mathcal{N}$ (and where we emphasize that the agents are within the same cone). All functions \tilde{w}_{ij} are assumed to satisfy the following condition,

$$\tilde{w}_{ij}(\nu_i, \nu_j) > 0 \forall (\nu_i, \nu_j) \in \mathcal{C}(\alpha, \bar{\nu})^2 \text{ with } \langle \nu_i, \nu_j \rangle \neq 1. \quad (9)$$

Thus, from continuity, it follows that the weight between two neighbors is zero if and only if they are synchronized, though the weight may be arbitrarily small when the neighbors are arbitrarily close to each other or when the neighbors are close the boundaries of the domain of the weight functions.

The dependency of the dynamics (7)-(8) on time comes from the time varying network graph, and more specifically, the time varying neighbor set of each agent, as specified in (8).

Although the results of the paper remain valid under other types of switching (see proof of Theorem 1), in practical cases the switching instants of each agent's control law cannot accumulate to a certain time value. In order to formulate this observation as an assumption, we adopt Definition 2.3. in [32], and say that the switching signal σ has an average dwell-time $\tau_D > 0$ and a chatter bound $N_0 \in \mathbb{N}$ if the number of switching times of σ in any open finite interval $(t_1, t_2) \subset \mathbb{R}_{\geq 0}$ is upper bounded by $N_0 + \frac{t_2 - t_1}{\tau_D}$.

Assumption 2: For each agent $i \in \mathcal{N}$, the time-varying neighbor set $\mathbb{R}_+ \ni t \mapsto \mathcal{N}_i(\sigma(t)) \subset \mathcal{N}$ switches with an average dwell-time $\tau_D^i > 0$ and a chatter bound $N_0^i \in \mathbb{N}$.

As explained in more detail in the next sections, each agent $i \in \mathcal{N}$ is in charge of providing the input that follows from composing the proposed output feedback control laws with the measurements made at each time instant. Thus, requiring an agent's neighbor set to switch with an average dwell time guarantees that the agent's input does not experience infinite many discontinuities in any time interval

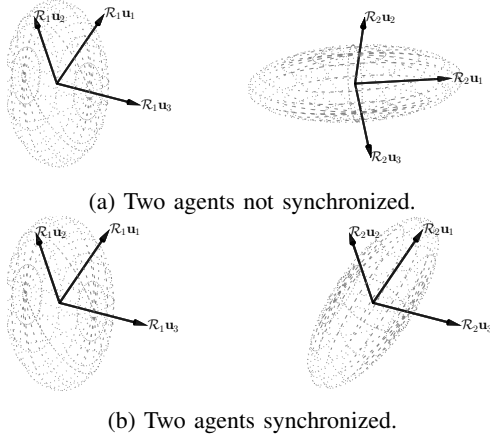


Fig. 2: In complete synchronization, N agents, indexed by $i = \{1, \dots, N\}$, synchronize their rotation matrices ($\mathbf{u}_1, \mathbf{u}_2$ and \mathbf{u}_3 stand for the canonical basis vectors in \mathbb{R}^3).

of finite length. In fact, if Assumption 2 is satisfied, then σ has an average dwell time, which allows us to invoke the results from Section III.

Proposition 3: If each agent's $i \in \mathcal{N}$ neighbor set $\mathbb{R}_+ \ni t \mapsto \mathcal{N}_i(\sigma(t))$ switches with an average dwell-time τ_D^i and chatter bound N_0^i , then the network dynamics (7) has a switching signal with average dwell time $\tau_D = \frac{1}{N} \min_{i \in \mathcal{N}} \tau_D^i$ and chatter bound $N_0 = N \max_{i \in \mathcal{N}} N_0^i$.

A proof is found in Appendix II.

A. Complete synchronization in $\mathbb{SO}(3)$ casted as synchronization in \mathbb{S}^3

Consider a group of N agents $\mathcal{R}_1, \dots, \mathcal{R}_N \in \mathbb{SO}(3) := \{\mathcal{R} \in \mathbb{R}^{3 \times 3} : \mathcal{R}^T \mathcal{R} = \mathcal{R} \mathcal{R}^T = \mathbf{I}, \det(\mathcal{R}) = 1\}$, where, for every $i \in N$, \mathcal{R}_i represents the orientation frame, w.r.t. an unknown inertial orientation frame, of agent i . We say that the agents are synchronized if they all share the same *complete* orientation, i.e., if $\mathcal{R}_1 = \dots = \mathcal{R}_N$, as illustrated in Fig. 2 for $N = 2$. The term *complete* synchronization is used in juxtaposition with *incomplete* synchronization as described in the next subsection. In incomplete synchronization, rather than synchronizing all three bodies axes, the agents synchronize only one body direction, and, as explained in the next section, complete synchronization does not guarantee incomplete synchronization (and vice-versa).

For each $i \in \mathcal{N}$, $\boldsymbol{\omega}_i : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}^3$ denotes the body-framed angular velocity of agent i , which can be actuated. Each rotation matrix $\mathcal{R}_i : \mathbb{R}_{\geq 0} \ni t \mapsto \mathcal{R}_i(t) \in \mathbb{SO}(3)$ evolves according to $\dot{\mathcal{R}}_i(t) = \mathbf{f}_{\mathcal{R}}(\mathcal{R}_i(t), \boldsymbol{\omega}_i(t))$ where

$$\begin{aligned} \mathbf{f}_{\mathcal{R}} : \mathbb{SO}(3) \times \mathbb{R}^3 \ni (\mathcal{R}, \boldsymbol{\omega}) &\mapsto \mathbf{f}_{\mathcal{R}}(\mathcal{R}, \boldsymbol{\omega}) \in T_{\mathcal{R}}\mathbb{SO}(3) \\ \mathbf{f}_{\mathcal{R}}(\mathcal{R}, \boldsymbol{\omega}) &:= \mathcal{R} \mathcal{S}(\boldsymbol{\omega}). \end{aligned} \quad (10)$$

If, at a time instant $t \in \mathbb{R}_{\geq 0}$, agent $i \in \mathcal{N}$ is aware of and can measure the relative attitude between itself and another agent j , then $j \in \mathcal{N}_i(\sigma(t))$. This motivates the definition of the measurement function $\mathbf{h}_i(t, \cdot)$ for each time instant

$t \in \mathbb{R}_{\geq 0}$, namely

$$\begin{aligned} \mathbf{h}_i(t, \cdot) : \mathbb{SO}(3)^N \ni \mathcal{R} &\mapsto \mathbf{h}_i(t, \mathcal{R}) \in \mathbb{SO}(3)^{|\mathcal{N}_i(\sigma(t))|} \\ \mathbf{h}_i(t, \mathcal{R}) &:= (\mathbf{h}_{ij_1}(\mathcal{R}), \dots, \mathbf{h}_{ij_{|\mathcal{N}_i(\sigma(t))|}}(\mathcal{R})), \end{aligned} \quad (11)$$

where $\{j_1, \dots, j_{|\mathcal{N}_i(\sigma(t))|}\} \equiv \mathcal{N}_i(\sigma(t))$ and where

$$\begin{aligned} \mathbf{h}_{ij} : \mathbb{SO}(3)^N \ni \mathcal{R} &:= (\mathcal{R}_1, \dots, \mathcal{R}_N) \mapsto \mathbf{h}_{ij}(\mathcal{R}) \in \mathbb{SO}(3) \\ \mathbf{h}_{ij}(\mathcal{R}) &:= \mathcal{R}_i^T \mathcal{R}_j, \end{aligned} \quad (12)$$

for each $j \in \mathcal{N}_i(\sigma(t))$. Thus, at each time instant $t \in \mathbb{R}_{\geq 0}$, agent $i \in \mathcal{N}$ measures the $|\mathcal{N}_i(\sigma(t))|$ rotation matrices corresponding to its neighbors orientation with respect to its own orientation. We emphasize that the measurement in (12) does not require an agent to be aware of its own rotation matrix or its neighbors rotation matrices (recall that these are specified in an unknown inertial orientation frame); rather it requires an agent to measure the projection of each of its neighbors three axes onto its own three axes.

Problem 1: For each agent $i \in \mathcal{N}$ and time instant $t \geq 0$, given the measurement function (11), design time-varying decentralized feedback laws $\boldsymbol{\omega}_i^h(t, \cdot) : \mathbb{SO}(3)^{|\mathcal{N}_i(\sigma(t))|} \mapsto \mathbb{R}^3$, such that asymptotic synchronization of $\mathcal{R} := (\mathcal{R}_1, \dots, \mathcal{R}_N) : \mathbb{R}_{\geq 0} \mapsto \mathbb{SO}(3)^N$ is attained, where $\dot{\mathcal{R}}_i(t) = \mathbf{f}_{\mathcal{R}}(\mathcal{R}_i(t), \boldsymbol{\omega}_i^h(t, \mathbf{h}_i(t, \mathcal{R}(t))))$ for every $i \in \mathcal{N}$.

Problem 1 may be restated as finding a control law for each agent that depends exclusively on the measurement function as defined in (11), and which encodes the partial state information available to each agent at a given time instant.

Definition 2: The angular displacement between two rotation matrices is given by $\theta : \mathbb{SO}(3) \times \mathbb{SO}(3) \ni (\mathcal{R}_1, \mathcal{R}_2) \mapsto \theta(\mathcal{R}_1, \mathcal{R}_2) := \arccos\left(\frac{\text{tr}(\mathcal{R}_1^T \mathcal{R}_2) - 1}{2}\right) \in [0, \pi]$.

For each agent $i \in \mathcal{N}$ and each time instant $t \in \mathbb{R}_{\geq 0}$, consider the control law $\boldsymbol{\omega}_i^h(t, \cdot) : \mathbb{SO}(3)^{|\mathcal{N}_i(\sigma(t))|} \ni \mathbf{h}_i := (\mathbf{h}_{ij_1}, \dots, \mathbf{h}_{ij_{|\mathcal{N}_i(\sigma(t))|}}) \mapsto \boldsymbol{\omega}_i^h(t, \mathbf{h}_i) \in \mathbb{R}^3$ defined as

$$\boldsymbol{\omega}_i^h(t, \mathbf{h}_i) := \sum_{j \in \mathcal{N}_i(\sigma(t))} w_{ij}(\theta(\mathbf{I}, \mathbf{h}_{ij})) \mathcal{S}^{-1}\left(\frac{\mathbf{h}_{ij} - \mathbf{h}_{ij}^T}{2}\right) \quad (13)$$

where $w_{ij} : [0, \pi] \mapsto \mathbb{R}_{\geq 0}$ is continuous and satisfies

$$w_{ij}(\theta) > 0 \quad \forall \theta \in (0, \pi]. \quad (14)$$

Notice that w_{ij} corresponds to a weight on the feedback law (13) that agent i assigns to the displacement between itself and its neighbor j . Denote $\boldsymbol{\omega}_i^{cl}$ as the composition of the output feedback control law (13) with the output function (11), i.e., $\boldsymbol{\omega}_i^{cl} : \mathbb{R}_+ \times \mathbb{SO}(3)^N \ni (t, \mathcal{R}) \mapsto \boldsymbol{\omega}_i^{cl}(t, \mathcal{R}) := \boldsymbol{\omega}_i^h(t, \mathbf{h}_i(t, \mathcal{R})) \in \mathbb{R}^3$. It follows that

$$\boldsymbol{\omega}_i^{cl}(t, \mathcal{R}) = \sum_{j \in \mathcal{N}_i(\sigma(t))} w_{ij}(\theta(\mathbf{I}, \mathcal{R}_i^T \mathcal{R}_j)) \mathcal{S}^{-1}\left(\frac{\mathcal{R}_i^T \mathcal{R}_j - \mathcal{R}_j^T \mathcal{R}_i}{2}\right) \quad (15)$$

where we have made use of the notation $\mathcal{R} = (\mathcal{R}_1, \dots, \mathcal{R}_N)$. We emphasize that if Assumption 2 is satisfied, then (13) (and (15)) does not have infinite many discontinuities in any time interval of finite length, which in turn implies that it can be implemented in a practical scenario.

Recall that we wish to analyze different problems under

a common framework where agents are unit vectors. Thus, in order to cast complete synchronization in $\mathbb{SO}(3)$ in the form (7), we perform a change of coordinates based on unit quaternions. This change of variables serves only the purpose of analysis, while the implemented control law is still that in (13).

For that purpose, and for convenience, denote

$$\tilde{\mathbb{SO}}(3) := \{\mathcal{R} \in \mathbb{SO}(3) : \theta(\mathbf{I}, \mathcal{R}) < \pi\}$$

which is an open subset of $\mathbb{SO}(3)$, and where $\theta(\mathbf{I}, \mathcal{R}) < \pi \Leftrightarrow 1 + \text{tr}(\mathcal{R}) > 0$ (see Definition 2). Consider then the map $T_r^q : \tilde{\mathbb{SO}}(3) \ni \mathcal{R} \mapsto T_r^q(\mathcal{R}) \in \mathbb{S}^3$ defined as

$$T_r^q(\mathcal{R}) := \frac{1}{2} \left(\sqrt{1 + \text{tr}(\mathcal{R})}, \frac{\mathcal{S}^{-1}(\mathcal{R} - \mathcal{R}^T)}{\sqrt{1 + \text{tr}(\mathcal{R})}} \right), \quad (16)$$

with T_r^q smooth on $\tilde{\mathbb{SO}}(3)$. Intuitively, T_r^q is a map that transforms a rotation matrix into a unit vector in \mathbb{R}^4 , also named unit quaternion, whose first component is positive. The idea followed later is: (i) given a rotation matrix $\mathcal{R} \in \tilde{\mathbb{SO}}(3)$, to consider the quaternion $\mathbf{q} = T_r^q(\mathcal{R})$; and, (ii) given the closed loop dynamics of the rotation matrix, to compute the closed loop dynamics of the quaternion, which are in the common form (8). For that purpose, consider also the map $T_q^r : \mathbb{S}^3 \ni \mathbf{q} := (\bar{q}, \hat{\mathbf{q}}) \mapsto T_q^r(\mathbf{q}) \in \mathbb{SO}(3)$ defined as

$$T_q^r(\mathbf{q}) := \mathbf{I} + 2\bar{q}\mathcal{S}(\hat{\mathbf{q}}) + 2\mathcal{S}(\hat{\mathbf{q}})\mathcal{S}(\hat{\mathbf{q}}),$$

where T_q^r restricted to the image of T_r^q yields the inverse map of T_r^q , i.e., $T_q^r \circ T_r^q = \text{id}_{\tilde{\mathbb{SO}}(3)}$.

Let us now compute the kinematics of a quaternion computed from the mapping (16). Given $\mathcal{R} \in \mathbb{SO}(3)$, consider $\mathbf{q} = T_r^q(\mathcal{R})$. Given the kinematics (10), one can compute the kinematics of the unit quaternion, as done in Appendix IV. It follows that

$$\dot{\mathbf{q}} \equiv \mathbf{f}_q : \tilde{\mathbb{S}}^3 \times \mathbb{R}^3 \ni (\mathbf{q}, \boldsymbol{\omega}) \mapsto \mathbf{f}_q(\mathbf{q}, \boldsymbol{\omega}) \in T_{\mathbf{q}}\tilde{\mathbb{S}}^3$$

is given by

$$\dot{\mathbf{q}}(\mathbf{q}, \boldsymbol{\omega}) \equiv \mathbf{f}_q(\mathbf{q}, \boldsymbol{\omega}) = \frac{1}{2}Q(\mathbf{q})[\mathbf{0}_{3 \times 1} \ \mathbf{I}_3]^T \boldsymbol{\omega}, \quad (17)$$

where, given any $\mathbf{q} \in \mathbb{S}^3$, $Q(\mathbf{q}) \in T_{\mathbf{q}}\tilde{\mathbb{S}}^3 \subset \mathbb{R}^{4 \times 4}$ is the linear map that satisfies

$$Q(\mathbf{q})\mathbf{v} := \langle \mathbf{q}, \mathbf{e}_1 \rangle \mathbf{v} - \langle \mathbf{q}, \mathbf{v} \rangle \mathbf{e}_1 + \langle \mathbf{e}_1, \mathbf{v} \rangle \mathbf{q} + \text{diag}(0, \mathcal{S}(\hat{\mathbf{q}}))\mathbf{v}, \quad (18)$$

for any $\mathbf{v} \in \mathbb{R}^4$ (it is easy to verify that $\langle \mathbf{q}, Q(\mathbf{q})\mathbf{v} \rangle = 0$).

The control law (15) may also be rewritten in terms of unit quaternions, which motivates the definition of $\boldsymbol{\omega}_i^q(t, \cdot) : (\tilde{\mathbb{S}}^3)^N \ni \mathbf{q} := (\mathbf{q}_1, \dots, \mathbf{q}_N) \mapsto \boldsymbol{\omega}_i^q(t, \mathbf{q}) \in \mathbb{R}^3$ as

$$\begin{aligned} \boldsymbol{\omega}_i^q(t, \mathbf{q}) &:= \boldsymbol{\omega}_i^{cl}(t, \mathcal{R})|_{\mathcal{R}=T_r^N(\mathbf{q})} \\ &= \sum_{j \in \mathcal{N}_i(\sigma(t))} 2 \langle \mathbf{q}_i, \mathbf{q}_j \rangle w_{ij}(\theta(\mathbf{q}_i, \mathbf{q}_j)) [\mathbf{0}_{3 \times 1} \ \mathbf{I}] Q(\mathbf{q}_i)^T \mathbf{q}_j \end{aligned} \quad (19)$$

where we have made use of (47) and (48) in Appendix IV, and where, for brevity, we denoted $\theta(\mathbf{q}_i, \mathbf{q}_j) := \arccos(2 \langle \mathbf{q}_i, \mathbf{q}_j \rangle^2 - 1)$. Denote $\tilde{w}_{ij}(\mathbf{q}_i, \mathbf{q}_j) :=$

$\langle \mathbf{q}_i, \mathbf{q}_j \rangle w_{ij}(\arccos(2 \langle \mathbf{q}_i, \mathbf{q}_j \rangle^2 - 1))$, and notice that this satisfies (9) for $\alpha = \frac{\pi}{4}$ and for any $\bar{\nu} \in \mathbb{S}^3$ (see also (14)).

Then, the unit quaternion kinematics (17), when composed with the control law (19), is in the same form as (8), i.e.

$$\begin{aligned} \mathbf{f}_q(\mathbf{q}_i, \boldsymbol{\omega}_i^q(t, \mathbf{q}))|_{\mathbf{q}=(\mathbf{q}_1, \dots, \mathbf{q}_N)} &= \\ (17) \quad & \frac{1}{2}Q(\mathbf{q}_i)[\mathbf{0}_{3 \times 1} \ \mathbf{I}_3]^T \boldsymbol{\omega}_i^q(t, \mathbf{q})|_{\mathbf{q}=(\mathbf{q}_1, \dots, \mathbf{q}_N)} \\ (19) \quad &= \sum_{j \in \mathcal{N}_i(\sigma(t))} \tilde{w}_{ij}(\mathbf{q}_i, \mathbf{q}_j) Q(\mathbf{q}_i) \text{diag}(0, \mathbf{I}_3) Q(\mathbf{q}_i)^T \mathbf{q}_j \\ (18) \quad &= \sum_{j \in \mathcal{N}_i(\sigma(t))} \tilde{w}_{ij}(\mathbf{q}_i, \mathbf{q}_j) \Pi(\mathbf{q}_i) \mathbf{q}_j \\ (8) \quad &=: \tilde{\mathbf{f}}_{i, \sigma(t)}(\mathbf{q}). \end{aligned}$$

We have then casted this problem in the form (7) with $\nu \equiv \mathbf{q} \in (\mathbb{S}^3)^N$.

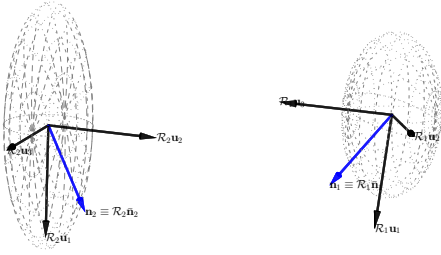
Remark 4: Consider $(\mathbf{q}_1, \dots, \mathbf{q}_N) \in \mathcal{C}(\frac{\pi}{4}, \bar{\mathbf{q}})^N$ for some $\bar{\mathbf{q}} \in \mathbb{S}^3$, as required by \tilde{w}_{ij} to be positive. Without loss of generality, one may assume that $\bar{\mathbf{q}} = \mathbf{e}_1$, where $T_r^q(\bar{\mathbf{q}}) = \mathbf{I}$. It then follows that $\theta(T_r^q(\bar{\mathbf{q}}), T_r^q(\mathbf{q}_i)) = \theta(\mathbf{I}, T_r^q(\mathbf{q}_i)) = \arccos(2 \langle \bar{\mathbf{q}}, \mathbf{q}_i \rangle^2 - 1) \leq \arccos(2 \cos^2(\frac{\pi}{4}) - 1) = \frac{\pi}{2}$ for all $i \in \mathcal{N}$. As such, i.e. all rotation matrices are $\frac{\pi}{2}$ close to the identity matrix, and therefore if $(\mathbf{q}_1, \dots, \mathbf{q}_N) \in \mathcal{C}(\frac{\pi}{4}, \bar{\mathbf{q}})^N$ then $(\mathcal{R}_1, \dots, \mathcal{R}_N) \in \tilde{\mathbb{SO}}(3)^N$.

B. Incomplete synchronization in $\mathbb{SO}(3)$ casted as synchronization in \mathbb{S}^2

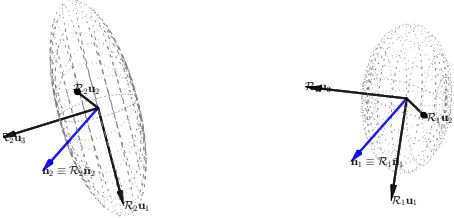
In this section, we consider again a group of N agents operating in $\mathbb{SO}(3)$, but with a different synchronization objective. As in Section IV-A, for each $i \in N$, \mathcal{R}_i represents the orientation frame of agent i . Additionally, for each agent i there is a constant body direction $\bar{\mathbf{n}}_i \in \mathbb{S}^2$, known by the agent and its neighbors, which is required to synchronize with all the other agents' body directions. The goal of incomplete attitude synchronization in $\mathbb{SO}(3)$ is that all agents share the same orientation along the chosen body directions; i.e., given $(\mathcal{R}_1, \dots, \mathcal{R}_N) \in \mathbb{SO}(3)^N$ and $(\bar{\mathbf{n}}_1, \dots, \bar{\mathbf{n}}_N) \in (\mathbb{S}^2)^N$, *incomplete* synchronization exists when $\mathcal{R}_1 \bar{\mathbf{n}}_1 = \dots = \mathcal{R}_N \bar{\mathbf{n}}_N$, as illustrated in Fig. 3 for $N = 2$. We note that the requirement for incomplete synchronization is independent from that of complete synchronization: i.e., complete synchronization does not imply incomplete synchronization (consider the case where $\bar{\mathbf{n}}_1 \neq \bar{\mathbf{n}}_2$), and vice-versa.

Similarly to Section IV-A, for each $i \in \mathcal{N}$, $\boldsymbol{\omega}_i : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}^3$ denotes the body-framed angular velocity of agent i , which can be actuated. Again, each rotation matrix $\mathcal{R}_i : \mathbb{R}_{\geq 0} \ni t \mapsto \mathcal{R}_i(t) \in \mathbb{SO}(3)$ evolves according to $\dot{\mathcal{R}}_i(t) = \mathbf{f}_{\mathcal{R}}(\mathcal{R}_i(t), \boldsymbol{\omega}_i(t))$ with $\mathbf{f}_{\mathcal{R}}$ as defined in (10). If, additionally, we consider some constant $\bar{\mathbf{n}}_i \in \mathbb{S}^2$, then $\mathbf{n}_i := \mathcal{R}_i \bar{\mathbf{n}}_i : \mathbb{R}_{\geq 0} \ni t \mapsto \mathbf{n}_i(t) \in \mathbb{S}^2$, evolves according to $\dot{\mathbf{n}}_i(t) = \mathbf{f}_{\mathbf{n}}(\mathcal{R}_i(t), \boldsymbol{\omega}_i(t), \bar{\mathbf{n}}_i)$ where

$$\begin{aligned} \mathbf{f}_{\mathbf{n}} : \mathbb{SO}(3) \times \mathbb{R}^3 \times \mathbb{S}^2 \ni (\mathcal{R}, \boldsymbol{\omega}, \bar{\mathbf{n}}) &\mapsto \mathbf{f}_{\mathbf{n}}(\mathcal{R}, \boldsymbol{\omega}, \bar{\mathbf{n}}) \in T_{\mathcal{R}\bar{\mathbf{n}}} \mathbb{S}^2 \\ \mathbf{f}_{\mathbf{n}}(\mathcal{R}, \boldsymbol{\omega}, \bar{\mathbf{n}}) &:= \mathbf{f}_{\mathcal{R}}(\mathcal{R}, \boldsymbol{\omega}) \bar{\mathbf{n}} = \mathcal{R} \mathcal{S}(\boldsymbol{\omega}) \bar{\mathbf{n}} = -\mathcal{S}(\mathcal{R} \bar{\mathbf{n}}) \mathcal{R} \boldsymbol{\omega} \end{aligned} \quad (20)$$



(a) Two agents not synchronized, i.e., $\mathbf{n}_1 \neq \mathbf{n}_2$ with $\bar{\mathbf{n}}_1 = \bar{\mathbf{n}}_2 = 3^{-\frac{1}{2}}[1 \ 1 \ 1]^T$.



(b) Two agents synchronized, i.e., $\mathbf{n}_1 = \mathbf{n}_2$ with $\bar{\mathbf{n}}_1 = \bar{\mathbf{n}}_2 = 3^{-\frac{1}{2}}[1 \ 1 \ 1]^T$.

Fig. 3: In incomplete synchronization, all agents $i = \{1, \dots, N\}$, align the unit vectors $\mathbf{n}_i \triangleq \mathcal{R}_i \bar{\mathbf{n}}_i$, where $\bar{\mathbf{n}}_i$ is fixed in rigid body i ($\mathbf{u}_1, \mathbf{u}_2$ and \mathbf{u}_3 stand for the canonical basis vectors in \mathbb{R}^3).

In what follows, we make use of the notation below,

$$\mathcal{R} = (\mathcal{R}_1, \dots, \mathcal{R}_N), \quad (21)$$

$$(\mathbf{n}_1, \dots, \mathbf{n}_N) = (\mathcal{R}_1 \bar{\mathbf{n}}_1, \dots, \mathcal{R}_N \bar{\mathbf{n}}_N). \quad (22)$$

If, at a time instant $t \in \mathbb{R}_{\geq 0}$, agent $i \in \mathcal{N}$ is aware of the relative attitude of agent's j unit vector to be synchronized, then $j \in \mathcal{N}_i(\sigma(t))$. This motivates the definition of the measurement function $\mathbf{h}_i(t, \cdot)$ for each time instant $t \in \mathbb{R}_{\geq 0}$, namely

$$\begin{aligned} \mathbf{h}_i(t, \cdot) : \mathbb{SO}(3)^N \ni \mathcal{R} &\mapsto \mathbf{h}_i(t, \mathcal{R}) \in (\mathbb{S}^2)^{|\mathcal{N}_i(\sigma(t))|} \\ \mathbf{h}_i(t, \mathcal{R}) &:= (\mathbf{h}_{ij_1}(\mathcal{R}), \dots, \mathbf{h}_{ij_{|\mathcal{N}_i(\sigma(t))|}}(\mathcal{R})), \end{aligned} \quad (23)$$

where $\{j_1, \dots, j_{|\mathcal{N}_i(\sigma(t))|}\} \equiv \mathcal{N}_i(\sigma(t))$ and where

$$\begin{aligned} \mathbf{h}_{ij} : \mathbb{SO}(3)^N \ni \mathcal{R} &\mapsto \mathbf{h}_{ij}(\mathcal{R}) \in \mathbb{S}^2 \\ \mathbf{h}_{ij}(\mathcal{R}) &:= \mathcal{R}_i^T \mathcal{R}_j \bar{\mathbf{n}}_j = \mathcal{R}_i^T \mathbf{n}_j, \end{aligned} \quad (24)$$

for each $j \in \mathcal{N}_i(\sigma(t))$, and where we have used of the notation in (21) and (22). Thus, at each time instant $t \in \mathbb{R}_{\geq 0}$, agent $i \in \mathcal{N}$ measures the $|\mathcal{N}_i(\sigma(t))|$ unit vectors corresponding to the projection of a neighbor's unit vector onto agent's i orientation frame.

Problem 2: For each agent $i \in \mathcal{N}$ and time instant $t \geq 0$, given the measurement function (23), design time-varying decentralized feedback laws $\omega_i^h(t, \cdot) : (\mathbb{S}^2)^{|\mathcal{N}_i(\sigma(t))|} \mapsto \mathbb{R}^3$, such that asymptotic synchronization of $(\mathcal{R}_1 \bar{\mathbf{n}}_1, \dots, \mathcal{R}_N \bar{\mathbf{n}}_N) : \mathbb{R}_{\geq 0} \mapsto (\mathbb{S}^2)^N$ is accomplished, where $\bar{\mathcal{R}}_i(t) = \mathbf{f}_{\mathcal{R}}(\mathcal{R}_i(t), \omega_i^h(t, \mathbf{h}_i(t, \mathcal{R}(t))))$ for every $i \in \mathcal{N}$.

Problem 2 may be restated as finding a control law for

each agent that depends exclusively on the measurement function as defined in (23), and which encodes the partial state information available to each agent at a given time instant.

Definition 3: The angular displacement between two unit vectors is given by $\theta : \mathbb{S}^2 \times \mathbb{S}^2 \ni (\mathbf{n}_1, \mathbf{n}_2) \mapsto \theta(\mathbf{n}_1, \mathbf{n}_2) := \arccos(\langle \mathbf{n}_1, \mathbf{n}_2 \rangle) \in [0, \pi]$.

For each $t \in \mathbb{R}_{\geq 0}$ and agent $i \in \mathcal{N}$, consider the control law $\omega_i^h(t, \cdot) : (\mathbb{S}^2)^{|\mathcal{N}_i(\sigma(t))|} \ni \mathbf{h}_i := (\mathbf{h}_{ij_1}, \dots, \mathbf{h}_{ij_{|\mathcal{N}_i(\sigma(t))|}}) \mapsto \omega_i^h(t, \mathbf{h}) \in \mathbb{R}^3$ defined as

$$\omega_i^h(t, \mathbf{h}) := \sum_{j \in \mathcal{N}_i(\sigma(t))} w_{ij}(\theta(\bar{\mathbf{n}}_i, \mathbf{h}_{ij})) \mathcal{S}(\bar{\mathbf{n}}_i) \mathbf{h}_{ij}, \quad (25)$$

where $w_{ij} : [0, \pi] \mapsto \mathbb{R}_{\geq 0}$ is a continuous function satisfying

$$w_{ij}(\theta) > 0, \forall \theta \in (0, \pi], \quad (26)$$

and corresponding to a weight on the feedback law (25) agent i assigns to the angular displacement between itself and its neighbor j . Denote ω_i^{cl} as the composition of the output feedback control law (25) with the output function (23), i.e., $\omega_i^{cl} : \mathbb{R}_+ \times \mathbb{SO}(3)^N \ni (t, \mathcal{R}) \mapsto \omega_i^{cl}(t, \mathcal{R}) := \omega_i^h(t, \mathbf{h}_i(t, \mathcal{R})) \in \mathbb{R}^3$. It follows that

$$\begin{aligned} \omega_i^{cl}(t, \mathcal{R}) &:= \omega_i^h(t, \mathbf{h}_i(t, \mathcal{R})) \\ &= \sum_{j \in \mathcal{N}_i(\sigma(t))} w_{ij}(\theta(\bar{\mathbf{n}}_i, \mathcal{R}_i^T \mathcal{R}_j \bar{\mathbf{n}}_j)) \mathcal{S}(\bar{\mathbf{n}}_i) \mathcal{R}_i^T \mathcal{R}_j \bar{\mathbf{n}}_j \\ &= \mathcal{R}_i^T \sum_{j \in \mathcal{N}_i(\sigma(t))} w_{ij}(\theta(\mathbf{n}_i, \mathbf{n}_j)) \mathcal{S}(\mathbf{n}_i) \mathbf{n}_j, \end{aligned} \quad (27)$$

where we have used of (21) and (22); and of the fact that $\theta(\mathbf{n}_i, \mathbf{n}_j) = \theta(\mathcal{R}_i^T \mathbf{n}_i, \mathcal{R}_i^T \mathbf{n}_j)$ for any $\mathcal{R}_i \in \mathbb{SO}(3)$ and any $\mathbf{n}_i, \mathbf{n}_j \in \mathbb{S}^2$. We emphasize that the control law (27) is based on the output feedback control law, and thus depends only on the relative orientation measurements (see (23) and (24)); moreover, (27) is orthogonal to $\bar{\mathbf{n}}_i$, which implies that full angular velocity control is not necessary, i.e. we only need to control the angular velocity along the two directions orthogonal to $\bar{\mathbf{n}}_i$ ($\omega_i^h : \mathbb{R}_+ \times (\mathbb{S}^2)^{|\mathcal{N}_i(\sigma(t))|} \mapsto T_{\bar{\mathbf{n}}_i} \mathbb{S}^2 \subset \mathbb{R}^3$).

Denote $\tilde{w}_{ij}(\mathbf{n}_i, \mathbf{n}_j) = w_{ij}(\theta(\mathbf{n}_i, \mathbf{n}_j))$, which satisfies (9) for any $\alpha \in [0, \pi]$ and $\bar{\nu} \in \mathbb{S}^2$, due to (26) and Definition 3. With the above in mind, the kinematics (20), when composed with the proposed law (27) yield (8), i.e.,

$$\begin{aligned} \dot{\mathbf{f}}_n(\mathcal{R}_i, \omega_i^{cl}(t, (\mathcal{R}_1, \dots, \mathcal{R}_N)), \bar{\mathbf{n}}_i) &|_{\mathcal{R}_i \bar{\mathbf{n}}_i = \mathbf{n}_i \forall i \in \mathcal{N}} \\ &\stackrel{(20)}{=} -\mathcal{S}(\mathbf{n}_i) \mathcal{R}_i \omega_i^{cl}(t, (\mathcal{R}_1, \dots, \mathcal{R}_N)) |_{\mathcal{R}_i \bar{\mathbf{n}}_i = \mathbf{n}_i \forall i \in \mathcal{N}} \\ &\stackrel{(27)}{=} -\sum_{j \in \mathcal{N}_i(\sigma(t))} \tilde{w}_{ij}(\mathbf{n}_i, \mathbf{n}_j) \mathcal{S}(\mathbf{n}_i) \mathcal{S}(\mathbf{n}_i) \mathbf{n}_j \\ &= \sum_{j \in \mathcal{N}_i(\sigma(t))} \tilde{w}_{ij}(\mathbf{n}_i, \mathbf{n}_j) \Pi(\mathbf{n}_i) \mathbf{n}_j \\ &\stackrel{(8)}{=} \tilde{\mathbf{f}}_{i, \sigma(t)}(\mathbf{n}) |_{\mathbf{n}=(\mathbf{n}_1, \dots, \mathbf{n}_N) \in (\mathbb{S}^2)^N}. \end{aligned}$$

We have thus casted this problem in the form (7)-(8) with $\nu \equiv \mathbf{n} \in (\mathbb{S}^2)^N$.

As stated in Section IV, consensus in \mathbb{R}^n can also be casted as a synchronization problem in \mathbb{S}^n , for any $n \in \mathbb{N}$. This case is presented in Appendix V.

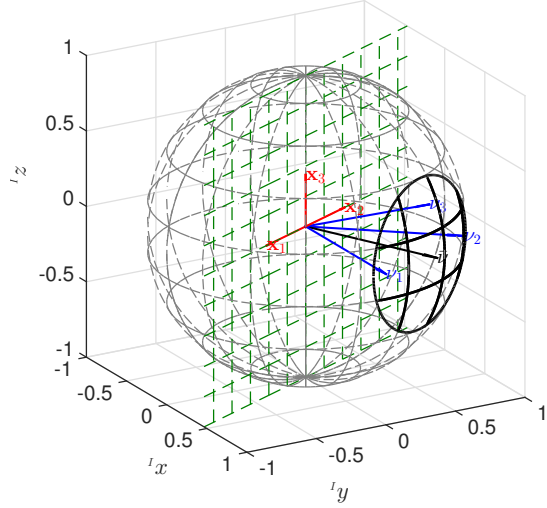


Fig. 4: Illustration of (28) as in Definition (4): $\mathbf{x}_i = \mathbf{h}_v(\nu_i) \stackrel{(28)}{=} Q_v^T \nu_i$, and green lines are those spanned by the columns of Q_v (which form a plane).

V. ANALYSIS

In this section, we analyze the solutions of (7)-(8), and show that given a wide set of initial conditions, asymptotic synchronization is guaranteed. Specifically, asymptotic synchronization is guaranteed if all unit vectors are initially contained in an open α^* -cone, i.e. if $\exists \bar{\nu} \in \mathbb{S}^n : \nu(0) \in \mathcal{C}(\alpha^*, \bar{\nu})^N$, where $\alpha^* = \frac{\pi}{2}$ for synchronization in \mathbb{S}^2 and consensus in \mathbb{R}^n , and $\alpha^* = \frac{\pi}{4}$ for synchronization in $\mathbb{SO}(3)$.

Remark 5: If $\nu(0) \in \mathcal{C}(\alpha, \bar{\nu})^N$ for some $\alpha \in [0, \alpha^*)$ and some $\bar{\nu} \in \mathbb{S}^n$, then Proposition 11 in Appendix III guarantees that there exist $n+1$ linearly independent unit vectors $\{\bar{\nu}_k \in \mathbb{S}^n\}_{k \in \{1, \dots, n+1\}}$ such $\nu(0) \in \mathcal{C}(\alpha^*, \bar{\nu}_k)^N \forall k \in \{1, \dots, n+1\}$. Thus, if $\nu(0)$ is contained in an open α cone, then there exist other bigger cones that contain $\nu(0)$; as such, the choice of $\bar{\nu}$ is not unique.

Next, we introduce a coordinate transformation that we exploit in order to cast the dynamics (8) into a form that satisfies the conditions of Theorem 1. In particular, given some $\bar{\nu} \in \mathbb{S}^2$ we consider the projection of the cone $\mathcal{C}(\frac{\pi}{2}, \bar{\nu})$ to the plane in \mathbb{R}^{n+1} orthogonal to $\bar{\nu}$ and containing zero, and then map this plane isometrically to \mathbb{R}^n .

Definition 4: Let $\bar{\nu} \in \mathbb{S}^n$ and $Q_v \in \mathbb{R}^{(n+1) \times n}$, such that $\bar{\nu}$ and the columns of Q_v form an orthonormal basis of \mathbb{R}^{n+1} , and consider the diffeomorphism $\mathbf{h}_v : \mathcal{C}(\frac{\pi}{2}, \bar{\nu}) \mapsto \mathcal{B}(1)$ (see Notation for the definition of \mathcal{B}), defined as

$$\mathbf{h}_v(\nu_i) = Q_v^T \nu_i. \quad (28)$$

Its derivative is given by $d\mathbf{h}_v = Q_v^T$, and its inverse $\mathbf{h}_v^{-1} : \mathcal{B}(1) \mapsto \mathcal{C}(\frac{\pi}{2}, \bar{\nu})$ is given by $\mathbf{h}_v^{-1}(\mathbf{x}_i) = \sqrt{1 - \|\mathbf{x}_i\|^2} \bar{\nu} + Q_v \mathbf{x}_i$.

Figure 4 illustrates the mapping \mathbf{h}_v as introduced in Definition 4, for $n = 2$ and $N = 3$.

Proposition 6: Consider $\bar{\nu} \in \mathbb{S}^n$ and $\nu_i, \nu_j \in \mathcal{C}(\frac{\pi}{2}, \bar{\nu})$. Then, $\langle \bar{\nu}, \nu \rangle = \sqrt{1 - \|\mathbf{h}_v(\nu)\|^2} > 0$ and the following

implications hold: $\|\mathbf{h}_v(\nu_i)\| > \|\mathbf{h}_v(\nu_j)\| \Leftrightarrow 0 < \langle \bar{\nu}, \nu_i \rangle < \langle \bar{\nu}, \nu_j \rangle$, and $\|\mathbf{h}_v(\nu_i)\| \geq \|\mathbf{h}_v(\nu_j)\| \Leftrightarrow 0 < \langle \bar{\nu}, \nu_i \rangle \leq \langle \bar{\nu}, \nu_j \rangle$.

Proof: Since $\mathbf{h}_v(\nu) = Q_v^T \nu$, then $\|\mathbf{h}_v(\nu)\|^2 = \langle Q_v^T \nu, Q_v^T \nu \rangle = \langle \nu, \Pi(\bar{\nu}) \nu \rangle = 1 - (\langle \bar{\nu}, \nu \rangle)^2$ (notice that $Q_v Q_v^T = \Pi(\bar{\nu})$). Since $\langle \bar{\nu}, \nu \rangle > 0$, for any $\nu \in \mathcal{C}(\frac{\pi}{2}, \bar{\nu})$, it follows that $\langle \bar{\nu}, \nu \rangle = \sqrt{1 - \|\mathbf{h}_v(\nu)\|^2}$. The implications in the Proposition follow since $\langle \bar{\nu}, \nu \rangle = \sqrt{1 - \|\mathbf{h}_v(\nu)\|^2}$ is decreasing with $\|\mathbf{h}_v(\nu)\|$, and since $\nu_i, \nu_j \in \mathcal{C}(\frac{\pi}{2}, \bar{\nu})$. ■

Consider now the solution $\nu : \mathbb{R}_{\geq 0} \mapsto (\mathbb{S}^n)^N$ of (7)-(8) with $\nu(0) \in \mathcal{C}(\frac{\pi}{2}, \bar{\nu})^N$ for some $\bar{\nu} \in \mathbb{S}^n$, which, as will be shown in Theorem 9, remains in $\mathcal{C}(\frac{\pi}{2}, \bar{\nu})^N$ for all $t \geq 0$; and define $\mathbf{x} : \mathbb{R}_{\geq 0} \mapsto (\mathbb{R}^n)^N$ as $\mathbf{x} = \mathbf{h}_v^N \circ \nu$. Then, based on the transformation introduced in Definition 4, it follows that $\dot{\mathbf{x}}(t) = \mathbf{f}_{\sigma(t)}(\mathbf{x}(t))$, where

$$\mathbf{f}_{\sigma(t)}(\mathbf{x}) = d\mathbf{h}_v^N(\nu) \tilde{\mathbf{f}}_{\sigma(t)}(\nu)|_{\nu=(\mathbf{h}_v^{-1})^N(\mathbf{x})}, \quad (29)$$

and $\dot{\mathbf{x}}_i = \mathbf{h}_v \circ \nu_i$ evolves according to $\dot{\mathbf{x}}_i(t) = \mathbf{f}_{i, \sigma(t)}(\mathbf{x}(t))$, where

$$\mathbf{f}_{i, \sigma(t)}(\mathbf{x}) = d\mathbf{h}_v(\nu_i) \tilde{\mathbf{f}}_{\sigma(t)}(\nu)|_{\nu=(\mathbf{h}_v^{-1})^N(\mathbf{x})} \quad (30)$$

$$\stackrel{(8) \text{ and Def 4}}{=} Q_v^T \sum_{j \in \mathcal{N}_i(\sigma(t))} \tilde{w}_{ij}(\nu_i, \nu_j) \Pi(\nu_i) \nu_j|_{\nu=(\mathbf{h}_v^{-1})^N(\mathbf{x})}.$$

It follows from (30) that, for $\nu \in \mathcal{C}(\frac{\pi}{2}, \bar{\nu})^N$ with $\bar{\nu} \in \mathbb{S}^n$, $\mathbf{x} = \mathbf{h}_v^N(\nu) \in \mathcal{B}(1)^N$ and any $p \in \mathcal{P}$,

$$\begin{aligned} \langle \mathbf{x}_i, \mathbf{f}_{i,p}(\mathbf{x}) \rangle &\stackrel{(30)}{=} \langle Q_v^T \nu_i, Q_v^T \sum_{j \in \mathcal{N}_i(p)} \tilde{w}_{ij}(\nu_i, \nu_j) \Pi(\nu_i) \nu_j \rangle \\ &= \langle \nu_i, \Pi(\bar{\nu}) \sum_{j \in \mathcal{N}_i(p)} \tilde{w}_{ij}(\nu_i, \nu_j) \Pi(\nu_i) \nu_j \rangle \\ &= \langle \nu_i - \langle \nu_i, \bar{\nu} \rangle \bar{\nu}, \sum_{j \in \mathcal{N}_i(p)} \tilde{w}_{ij}(\nu_i, \nu_j) \Pi(\nu_i) \nu_j \rangle \\ &= -\langle \bar{\nu}, \nu_i \rangle \sum_{j \in \mathcal{N}_i(p)} \tilde{w}_{ij}(\nu_i, \nu_j) \langle \bar{\nu}, \Pi(\nu_i) \nu_j \rangle \end{aligned} \quad (31)$$

The following result provides certain properties that are exploited in determining the sign of (31).

Proposition 7: Consider three unit vectors $\nu_1, \nu_2, \bar{\nu} \in \mathbb{S}^n$, satisfying $0 < \langle \bar{\nu}, \nu_1 \rangle \leq \langle \bar{\nu}, \nu_2 \rangle$. Then (a) $\langle \bar{\nu}, \Pi(\nu_1) \nu_2 \rangle = 0$ iff $\nu_1 = \nu_2$, and (b) $\langle \bar{\nu}, \Pi(\nu_1) \nu_2 \rangle > 0$ iff $\nu_2 \neq \nu_1$.

The proof is found in Appendix III. We show next, by combining Propositions 6 and 7 and exploiting (31), that the conditions of Theorem 1 are satisfied for the dynamics (29)-(30).

Proposition 8: Consider the vector field as defined in (29)-(30) for a certain $\bar{\nu} \in \mathbb{S}^n$ and assume that the switching signal $\sigma : \mathbb{R}_{\geq 0} \mapsto \mathcal{P}$ encodes only connected network graphs. Then the vector field (29)-(30) satisfies the conditions of Theorem 1 for $r = 1$.

Proof: In order to verify that the conditions of Theorem 1 are satisfied by the vector field in (29)-(30), we exploit (31) and the fact that for each $\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_N) \in \mathcal{B}(1)^N$ there exists a (unique) $\nu = (\nu_1, \dots, \nu_N) \in \mathcal{C}(\alpha^*, \bar{\nu})^N$ such that $\mathbf{x} = \mathbf{h}_v^N(\nu)$. We proceed with the verification of Condition 1) of Theorem 1 and pick $\mathbf{x} \in \mathcal{B}(1)^N$, where $\mathbf{x} = \mathbf{h}_v^N(\nu)$ for certain $\nu \in \mathcal{C}(\alpha^*, \bar{\nu})^N$. Notice, that since $\nu \in \mathcal{C}(\alpha^*, \bar{\nu})^N$,

it holds by definition that $\langle \bar{\nu}, \nu_i \rangle > \cos(\alpha^*) \geq 0 \forall i \in \mathcal{N}$. Therefore, since Condition 1a) depends exclusively on the sign of (31), we can ignore the effect of the positive term $\langle \bar{\nu}, \nu_i \rangle$.

In order to show Condition 1a), pick any $p \in \mathcal{P}$ and notice, that due to (9) and continuity of \tilde{w}_{ij} it holds that $\tilde{w}_{ij}(\nu_i, \nu_j) \geq 0$ for any $\nu_i, \nu_j \in \mathcal{C}(\alpha^*, \bar{\nu})$ and $i, j \in \mathcal{N}$. In addition, by recalling that $\mathcal{H}(\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_N)) = \arg \max_{i \in \mathcal{N}} \|\mathbf{x}_i\|$ and thus, that $\|\mathbf{x}_i\| \geq \|\mathbf{x}_j\|$ for all $i \in \mathcal{H}(\mathbf{x})$ and $j \in \mathcal{N}$, it follows from Proposition 6 that $\langle \bar{\nu}, \nu_i \rangle \leq \langle \bar{\nu}, \nu_j \rangle$ for all $j \in \mathcal{N}$. From the latter and the result of Proposition 7, we get that $\langle \bar{\nu}, \Pi(\nu_i) \nu_j \rangle \geq 0$ for all $j \in \mathcal{N}(p)$. Thus, we conclude from (31) that for any $p \in \mathcal{P}$ and $\mathbf{x} \in \mathcal{B}(1)^N$ it holds $\langle \mathbf{x}_i, \mathbf{f}_{i,p}(\mathbf{x}) \rangle \leq 0$, which by virtue of (5) implies that Condition 1a) is satisfied.

For the verification of Condition 1b), we additionally assume that $\mathbf{x} \notin \mathcal{C} := \{(\mathbf{x}_1, \dots, \mathbf{x}_N) \in (\mathbb{R}^n)^N : \mathbf{x}_1 = \dots = \mathbf{x}_N\}$. We will show that for each $p \in \mathcal{P}$ there exists $k \in \mathcal{H}(\mathbf{x})$ such that $\langle \mathbf{x}_k, \mathbf{f}_{k,p}(\mathbf{x}) \rangle < 0$. Indeed, suppose on the contrary that there exists $p \in \mathcal{P}$ such that

$$\langle \mathbf{x}_i, \mathbf{f}_{i,p}(\mathbf{x}) \rangle = 0 \forall i \in \mathcal{H}(\mathbf{x}), \quad (32)$$

and recall that the agents' network is not synchronized, since $\mathbf{x} \notin \mathcal{C}$. Consider then an $l \in \mathcal{H}(\mathbf{x})$, for which $\langle \mathbf{x}_l, \mathbf{f}_{l,p}(\mathbf{x}) \rangle = 0$ according to assumption (32). Notice also, that due to (9), Propositions 6 and 7 and (31), it can be shown (as in the proof of Condition 1a) above) that $\langle \mathbf{x}_l, \mathbf{f}_{l,p}(\mathbf{x}) \rangle = 0$ is satisfied only if all neighbors of agent l are synchronized with agent l , i.e., only if $\nu_j = \nu_l \Leftrightarrow \mathbf{x}_j = \mathbf{x}_l$ for all $j \in \mathcal{N}_l(p)$. This implies that all $j \in \mathcal{N}_l(p)$ are contained in $\mathcal{H}(\mathbf{x})$, i.e., $\mathcal{N}_l(p) \cup \{l\} \subseteq \mathcal{H}(\mathbf{x})$. As such, by assumption (32), $\langle \mathbf{x}_j, \mathbf{f}_{j,p}(\mathbf{x}) \rangle = 0$ for all $j \in \mathcal{N}_l(p)$, which means that the previous rationale is applicable for all $j \in \mathcal{N}_l(p)$, thus leading to the conclusion that all neighbors of all neighbors of agent l are necessarily synchronized with each other. Since the graph encoded by $p \in \mathcal{P}$ is connected, the previous rationale, applied $N - 1$ times, leads to the conclusion that all agents are synchronized. Since $\mathbf{x} \notin \mathcal{C}$, a contradiction has been reached, and therefore, for each $p \in \mathcal{P}$, there exists a $k \in \mathcal{H}(\mathbf{x})$ for which $\langle \mathbf{x}_k, \mathbf{f}_{k,p}(\mathbf{x}) \rangle < 0$, and therefore condition 1b) of Theorem 1 is satisfied.

Finally, let $\mathbf{x} \in \mathcal{C}$. Since $\mathbf{x}_1 = \dots = \mathbf{x}_N \Leftrightarrow \nu_1 = \dots = \nu_N$, it follows from Proposition 7 that $\langle \mathbf{x}_i, \mathbf{f}_{i,p}(\mathbf{x}) \rangle = 0$ for all $p \in \mathcal{P}$ and $i \in \mathcal{N}$. Thus, the second condition of Theorem 1 is also satisfied. ■

Theorem 9: Consider the solution $\nu : \mathbb{R}_{\geq 0} \mapsto (\mathbb{S}^n)^N$ of (7) with $\nu(0) \in \mathcal{C}(\alpha^*, \bar{\nu})^N$ for some $\bar{\nu} \in \mathbb{S}^n$. Then, for a network graph connected at all times, i) $\nu(t) \in \mathcal{C}(\alpha, \bar{\nu})^N$ for all $t \geq 0$, where $\alpha = \arccos(\max_{i \in \mathcal{N}} \langle \bar{\nu}, \nu_i \rangle(0)) \in [0, \alpha^*)$; ii) ν synchronizes asymptotically and $\lim_{t \rightarrow \infty} \bar{\nu} \nu_i(t)$ exists for all $i \in \mathcal{N}$; and iii) all unit vectors converge to a constant unit vector, i.e. $\exists \nu^* \in \mathbb{S}^n : \lim_{t \rightarrow \infty} \nu(t) \in \mathcal{C}(0, \nu^*)^N$.

Proof: Consider a solution ν of (7), and $\mathbf{x} = \mathbf{h}_\nu^N \circ \nu$. Since α^* is either $\frac{\pi}{2}$ or $\frac{\pi}{4}$, then $\nu(0)$ is within the domain of \mathbf{h}_ν^N and moreover $\bar{\mathcal{B}}(r_0)^N \subset \mathcal{B}(1)^N$ where $r_0 \stackrel{(28)}{=} \max_{i \in \mathcal{N}} \|\mathbf{Q}_\nu^T \nu_i(0)\| \stackrel{\text{Prop 6}}{=} \sqrt{1 - \max_{i \in \mathcal{N}} (\langle \bar{\nu}, \nu_i \rangle(0))^2} < 1$

(where the latter inequality follows from the fact that $\nu(0) \in \mathcal{C}(\alpha^*, \bar{\nu})^N$). From Proposition 8, the dynamics (29) satisfy Theorem's 1 conditions and therefore the set $\bar{\mathcal{B}}(r_0)^N$ is positively invariant for trajectories of $\dot{\mathbf{x}}(t) = \mathbf{f}_{\sigma(t)}(\mathbf{x}(t))$. This, in turn, implies that the set $\mathbf{H}_\nu^{-1}(\bar{\mathcal{B}}(r_0)^N) = \bar{\mathcal{C}}(\alpha, \bar{\nu})^N$, where $\alpha = \arccos(\max_{i \in \mathcal{N}} \langle \bar{\nu}, \nu_i \rangle(0)) \in [0, \alpha^*)$, is positively invariant for trajectories of $\dot{\nu}(t) = \tilde{\mathbf{f}}_{\sigma(t)}(\nu(t))$; i.e., all unit vectors are forever contained in the closed α -cone they start on. This suffices to conclude part i) in the Theorem.

Let us now focus on part ii) of the Theorem. From Proposition 8, the dynamics (29) satisfy Theorem's 1 conditions. It follows from Theorem 1 that $\lim_{t \rightarrow \infty} \mathbf{x}_i(t) - \mathbf{x}_j(t) = 0$ for all $i, j \in \mathcal{N}$, which implies that $\lim_{t \rightarrow \infty} \nu_i(t) - \nu_j(t) = 0$, for all $i, j \in \mathcal{N}$ (see Proposition 6). Moreover, it follows that the Lyapunov function in Theorem 1 converges to a constant, i.e., $\lim_{t \rightarrow \infty} V(\mathbf{x}(t)) = \lim_{t \rightarrow \infty} \max_{i \in \mathcal{N}} \frac{1}{2} \|\mathbf{x}_i(t)\|^2 = \lim_{t \rightarrow \infty} \frac{1}{2} \|\mathbf{x}_1(t)\|^2 = V^\infty$, for some constant $0 \leq V^\infty \leq V(0) < \frac{1}{2}$. From Proposition 6, it follows that $\lim_{t \rightarrow \infty} \bar{\nu} \nu_i(t) = \lim_{t \rightarrow \infty} \sqrt{1 - \|\mathbf{x}_i(t)\|^2} = \sqrt{1 - 2V^\infty}$.

We now prove part iii) of the Theorem. Since $\nu(0) \in \mathcal{C}(\alpha^*, \bar{\nu})^N$ for some $\bar{\nu} \in \mathbb{S}^n$, Proposition 11 guarantees that there exist $n + 1$ linearly independent unit vectors $\{\bar{\nu}_1, \dots, \bar{\nu}_{n+1}\}$ such that $\nu(\cdot) \in \mathcal{C}(\alpha^*, \bar{\nu}_k)^N$ for all $k \in \{1, \dots, n + 1\}$. From part ii) of this Theorem, it follows that, for each $k \in \{1, \dots, n + 1\}$, there exists a constant $V_k^\infty < \frac{1}{2}$ such that $\lim_{t \rightarrow \infty} \bar{\nu}_k^T \nu_1(t) = \sqrt{1 - 2V_k^\infty}$. Thus, it follows that $\lim_{t \rightarrow \infty} A \nu_1(t) = b \Leftrightarrow \lim_{t \rightarrow \infty} \nu_1(t) = A^{-1}b$, where $A^T = [\bar{\nu}_1 \dots \bar{\nu}_{n+1}]$ is non-singular, since $\bar{\nu}_1, \dots, \bar{\nu}_{n+1}$ are linearly independent, and $b^T = [\sqrt{1 - 2V_1^\infty} \dots \sqrt{1 - 2V_{n+1}^\infty}]$. Since synchronization is asymptotically reached, $\lim_{t \rightarrow \infty} \nu_i(t) = A^{-1}b$ for all $i \in \mathcal{N}$. ■

VI. SIMULATIONS

In this section, we present simulations that illustrate some of the results in the previous sections.

All simulations are provided for a network of six agents, i.e., $\mathcal{N} = \{1, \dots, 6\}$, whose network topology is presented in Fig. 6. The neighbor sets for all agents change in time: for agent 1, \mathcal{N}_1 alternates between $\{2\}$ and $\{2, 4\}$; for agent 2, \mathcal{N}_2 alternates between $\{3\}$ and $\{3, 6\}$; for agent 3, \mathcal{N}_3 alternates between $\{4\}$ and $\{4, 5\}$; for agent 4, \mathcal{N}_4 alternates between $\{5\}$ and $\{5, 1\}$; for agent 5, \mathcal{N}_5 alternates between $\{6\}$ and $\{6, 3\}$; for agent 6, \mathcal{N}_6 alternates between $\{1\}$ and $\{1, 2\}$. For these time-varying neighbor sets, the network graph is connected at all times.

The switching time instants for the neighbor set of each agent $i \in \mathcal{N}$ are those from the sequences $\mathcal{T}^i = \{\frac{1}{i} + ki\}_{k \in \mathbb{N}}$, which are shown in the time axes in Fig 5. For the same initial conditions, we perform two simulations, each one with different weight functions: for agents whose $i \in \mathcal{N}$ is even, $g_{ij}(\theta) = j$ for both simulations; for agents whose $i \in \mathcal{N}$ is odd, we consider $g_{ij}(\theta) = j(2 - \cos(\theta))$ for simulation 1 and $g_{ij}(\theta) = j(1 - \cos(\theta))$ for simulation 2. Notice that $g_{ij}(0) = j > 0$ for the former case, and $g_{ij}(0) = 0$ for the

latter case, which means odd agents penalize small errors differently between the two simulations.

In Figs. 5a-5f, six unit vectors are randomly initialized in an open $\frac{\pi}{2}$ -cone around $(1, 0, 0) \in \mathbb{S}^2$. In Figs. 5a and 5d, the trajectories of the unit vectors on the unit sphere are shown, and a visual inspection indicates convergence to a synchronized network. In Figs. 5b and 5e, the function $V(\mathbf{x}) = \max_{i \in \mathcal{N}} \frac{1}{2} \|\mathbf{x}_i\|^2$ – used in Theorem 9 – is provided, and we can verify that despite being non-smooth, it is almost always decreasing; notice that $V(\mathbf{x}(\cdot))$ converges to a constant which quantifies the asymptotic angular distance between all unit vectors and $(1, 0, 0)$. In Figs. 5c and 5f, the angular distance, i.e., $\theta(\cdot, \cdot)$ as in Definition 3, between some agents is presented, and it indicates convergence to a synchronized network. Notice that in simulation 1 convergence is quicker when compared to simulation 2. This is a consequence of choosing, for simulation 2, weight functions that are zero when two unit vectors are synchronized, i.e., $g_{ij}(0) = 0$ for i odd. This means that odd agents do not penalize the error between themselves and their neighbors *as much as* when they are *close*, and thus leading to a slow convergence to a synchronized network. In turn, even agents $i \in \mathcal{N}$ tend to converge to some odd agent, when $|\mathcal{N}_i(\cdot)| = 1$ (if $|\mathcal{N}_i(\cdot)| = 1$ then $\mathcal{N}_i(\cdot)$ is a set composed of one odd number); and tend to converge to somewhere in between their two neighbors when $|\mathcal{N}_i(\cdot)| = 2$. This explains the oscillatory behavior for simulation 2 in Figs. 5e and 5f.

In Figs. 5g-5l, six rotation matrices were randomly initialized such that $\theta(\mathbf{I}, \mathcal{R}_i) \leq \frac{\pi}{2}$ for all $i \in \mathcal{N}$. In Figs. 5g and 5j, the trajectories of the rotation matrices are shown on a sphere of π radius¹, and a visual inspection indicates convergence to a synchronized network. In Figs. 5h and 5k, the function $V(\mathbf{x}) = \max_{i \in \mathcal{N}} \frac{1}{2} \|\mathbf{x}_i\|^2$ – used in Theorem 9 – is provided, and we can verify that despite being non-smooth, it is almost always decreasing; notice that $V(\mathbf{x}(\cdot))$ converges to a constant which quantifies the asymptotic angular distance between all rotation matrices and \mathbf{I} (the rotation matrix that all rotation matrices start *close* to). In Figs. 5i and 5l, the angular distance, i.e., $\theta(\cdot, \cdot)$ as in Definition 2, between some agents is presented, and it indicates convergence to a synchronized network. Notice that in simulation 1 convergence is quicker when compared to simulation 2. The explanation for this behavior is the same as that provided before, and it is a consequence of choosing, for simulation 2, weight functions that are zero when two rotation matrices are synchronized, i.e., $g_{ij}(0) = 0$ for i odd. The oscillatory behavior for simulation 2 in Figs. 5k and 5l is also explained by the same reasoning described before.

VII. CONCLUSIONS

In this paper, we studied attitude synchronization in \mathbb{S}^2 and in $\mathbb{SO}(3)$, for a group of agents under connected network switching graphs. We proposed switching output feedback control laws for each agent's angular velocity, which are

¹For each rotation matrix \mathcal{R}_i , we plot $\theta_i \mathbf{n}_i$ where $\theta_i = \theta(\mathbf{I}, \mathcal{R}_i) \in [0, \pi]$ and $\mathbf{n}_i = \frac{1}{2 \sin(\theta_i)} \mathcal{S}^{-1}(\mathcal{R}_i - \mathcal{R}_i^T) \in \mathcal{S}^2$.

decentralized and do not require a common orientation frame among agents. Our main contribution lied in transforming those two problems into a common framework, where all agents dynamics are transformed into unit vectors' dynamics on a sphere of appropriate dimension. Convergence to a synchronized network was guaranteed for a wide range of initial conditions. Directions for future work include extending all results to agents controlled at the torque level, rather than the angular velocity level.

APPENDIX I PROOF OF THEOREM 1

Proof: [of Theorem 1] In what that follows, we invoke [36, Corollary 4.7]. We emphasize that the latter corollary requires persistent dwell time signals and Lyapunov functions to be continuously differentiable. However, an average dwell time signal is necessarily a persistent dwell time signal [30]. On the other hand, the latter corollary can be extended to continuous Lyapunov functions by replacing [36, Theorem 1] with [37, Corollary 4.4 b)] (and making use of the Clarke generalized derivative). Finally, [36] considers multiple Lyapunov functions, one for each switching signal mode, whilst in this proof we restrict ourselves to a common Lyapunov function. For brevity, in what follows, we denote

$$\mathbf{x} \mapsto f(\mathbf{x}) := (\mathbb{R}^n)^N \ni \mathbf{x} := (\mathbf{x}_1, \dots, \mathbf{x}_N) \mapsto f(\mathbf{x}).$$

Consider then any $p \in \mathcal{P}$ and the vector field (3), and denote

$$\mathbf{x} \mapsto v_{i,p}(\mathbf{x}) := \langle \mathbf{x}_i, \mathbf{f}_{i,p}(\mathbf{x}) \rangle \in \mathbb{R}, \forall i \in \mathcal{N} \quad (33)$$

which are all continuous functions. Consider then the continuous function

$$\mathbf{x} \mapsto V(\mathbf{x}) := \max_{i \in \mathcal{N}} \frac{1}{2} \|\mathbf{x}_i\|^2 \in \mathbb{R}_0^+, \quad (34)$$

whose generalized gradient (in the sense of Clarke) is given by (see (4) and denote $\text{co}(S)$ as the convex hull of a finite point set $S \subset \mathbb{R}^m$, for any $m \in \mathbb{N}$)

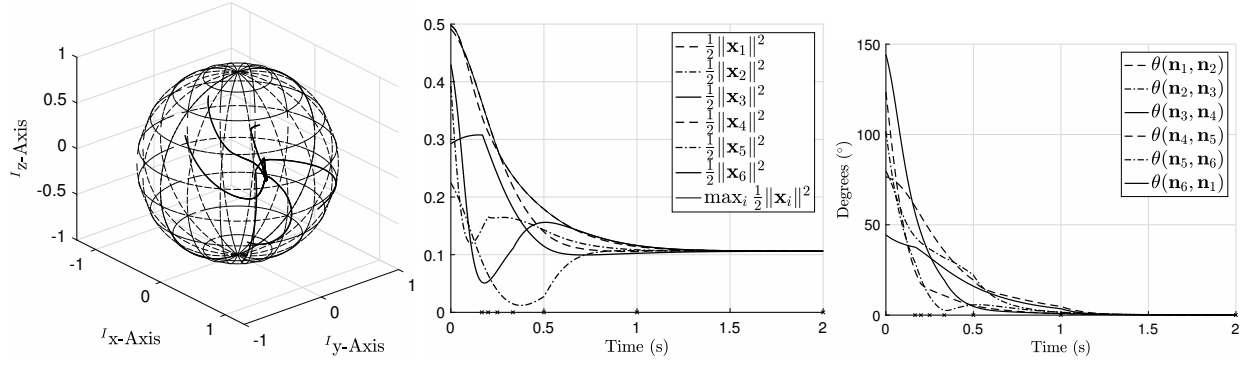
$$\mathbf{x} \mapsto \partial V(\mathbf{x}) = \text{co}(\{\mathbf{e}_i \otimes \mathbf{x}_i\}_{i \in \mathcal{H}(\mathbf{x})}) \subseteq \mathbb{R}^{nN}, \quad (35)$$

and where we emphasize that $\cup_{v \leq \frac{1}{2}r^2} V^{-1}(v) = (\bar{\mathcal{B}}(r))^N$ for any $r \in \mathbb{R}_+$ (see Notation for definition of $\bar{\mathcal{B}}$). The generalized directional derivative of V along (3), for a mode $p \in \mathcal{P}$, is then given by

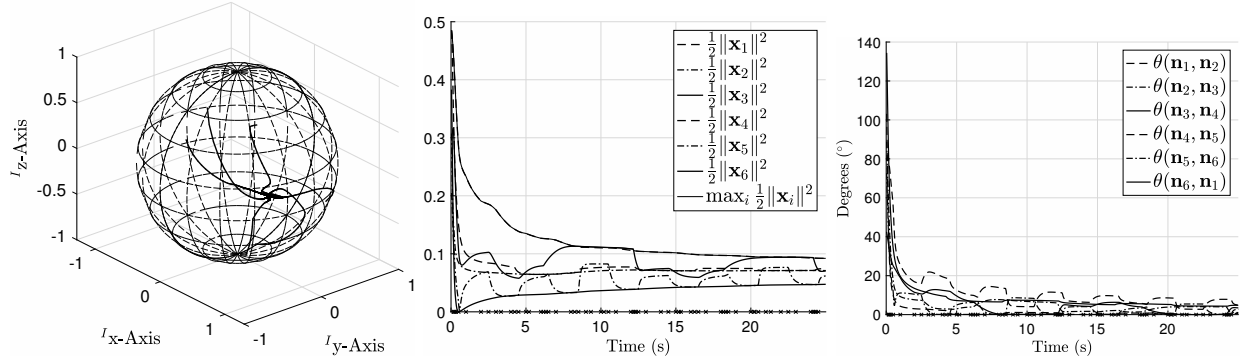
$$\begin{aligned} \mathbf{x} \mapsto V_p^\circ(\mathbf{x}) &:= \max_{dV \in \partial V(\mathbf{x})} \langle dV, \mathbf{f}_p(\mathbf{x}) \rangle \\ &= \max \text{co}(\{\langle \mathbf{x}_i, \mathbf{f}_{i,p}(\mathbf{x}) \rangle\}_{i \in \mathcal{H}(\mathbf{x})}). \end{aligned} \quad (36)$$

Recall (33), and notice that the Theorem's condition 1a) reads as $v_{i,p}(\mathbf{x})|_{\forall i \in \mathcal{H}(\mathbf{x})} \leq 0$ and condition 1b) reads as $\exists k \in \mathcal{H}(\mathbf{x}) : v_{i,p}(\mathbf{x})|_{i \in \mathcal{H}(\mathbf{x})} < 0$, for any $p \in \mathcal{P}$. As such, the generalized derivative in (36) can be expressed equivalently as

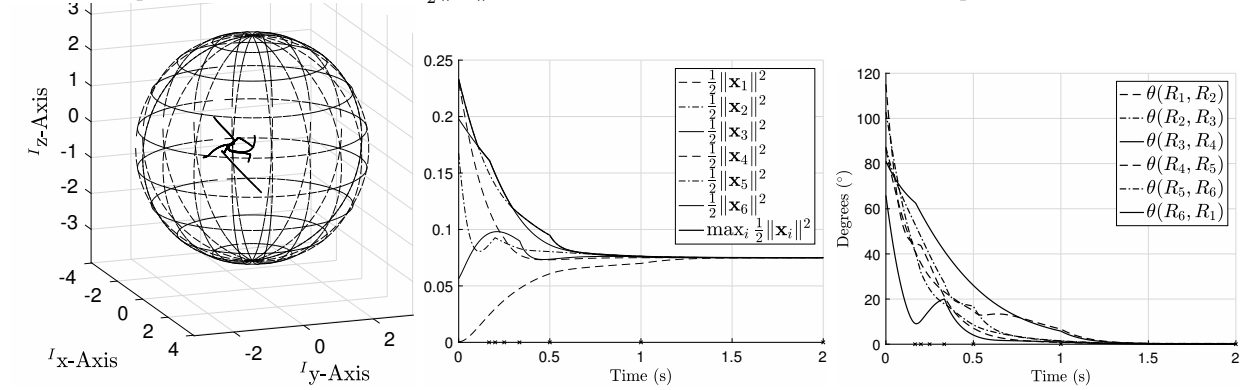
$$\mathbf{x} \mapsto V_p^\circ(\mathbf{x}) = \begin{cases} v_{i,p}(\mathbf{x})|_{\text{any } i \in \mathcal{H}(\mathbf{x})} < 0 & \mathbf{x} \in A_p \\ \max \text{co}(\{v_{i,p}(\mathbf{x})\}_{i \in \mathcal{H}(\mathbf{x})}) \leq 0 & \mathbf{x} \in B_p \\ 0 & \mathbf{x} \in \mathcal{C} \end{cases} \quad (37)$$



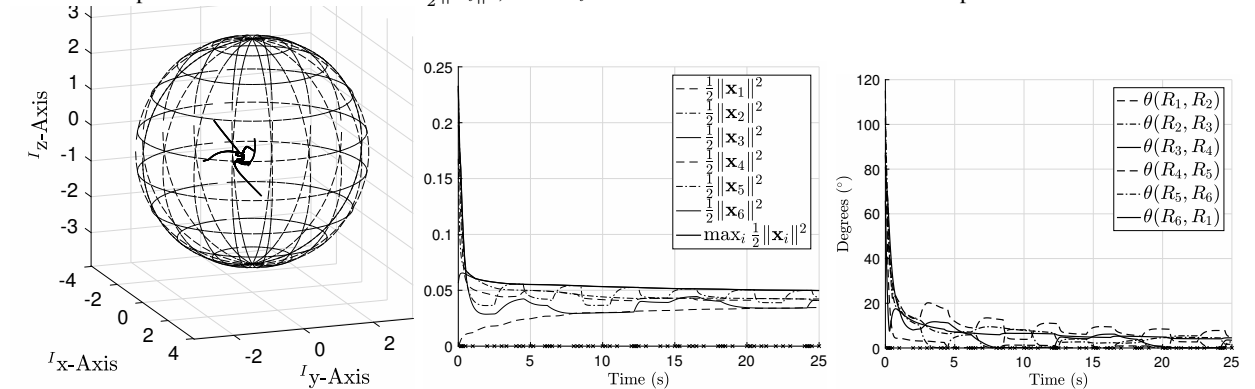
(a) Simulation 1: Trajectories of unit vectors in unit sphere (b) Simulation 1: time evolution of $\frac{1}{2}\|\mathbf{x}_i\|^2$, with \mathbf{x}_i as in Definition 4 (c) Simulation 1: angular distance between some pairs of unit vectors



(d) Simulation 2: Trajectories of unit vectors in unit sphere (e) Simulation 2: time evolution of $\frac{1}{2}\|\mathbf{x}_i\|^2$, with \mathbf{x}_i as in Definition 4 (f) Simulation 2: angular distance between some pairs of unit vectors



(g) Simulation 1: Trajectories of rotation matrices in sphere of π radius (h) Simulation 1: time evolution of $\frac{1}{2}\|\mathbf{x}_i\|^2$, with \mathbf{x}_i as in Definition 4 (i) Simulation 1: angular distance between some pairs of rotation matrices



(j) Simulation 2: Trajectories of rotation matrices in sphere of π radius (k) Simulation 2: time evolution of $\frac{1}{2}\|\mathbf{x}_i\|^2$, with \mathbf{x}_i as in Definition 4 (l) Simulation 2: angular distance between some pairs of rotation matrices

Fig. 5: Simulations 1 and 2.

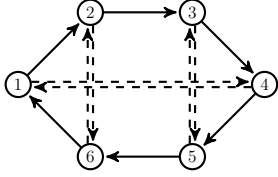


Fig. 6: Time-varying digraph with 6 agents.

where

$$\begin{aligned} A_p &:= \{\mathbf{x} \in \mathbb{R}^{nN} : \forall k, l \in H(\mathbf{x}), v_{k,p}(\mathbf{x}) = v_{l,p}(\mathbf{x}) < 0\} \\ B_p &:= \{\mathbf{x} \in \mathbb{R}^{nN} : \exists k, l \in H(\mathbf{x}), v_{k,p}(\mathbf{x}) \neq v_{l,p}(\mathbf{x})\}. \end{aligned} \quad (39)$$

The function V in (34) is lower bounded and its generalized derivative along (1) is non-positive. This implies that $\lim_{t \rightarrow \infty} V(\mathbf{x}(t)) =: V^\infty \in [0, V(\mathbf{x}(0))]$; and that $\bar{\mathcal{B}}(r_0)^N$, with $r_0 := \max_{i \in \mathcal{N}} \|\mathbf{x}_i(0)\|$, is positively invariant (since $V(\mathbf{x}) \leq V(\mathbf{x}(0)) \Leftrightarrow \mathbf{x} \in \cup_{r \leq V(\mathbf{x}(0))} V^{-1}(r) = \bar{\mathcal{B}}(r_0)^N$).

Moreover, it follows from (37), that

$$(V_p^\circ)^{-1}(0) \subseteq \mathcal{C} \cup B_p,$$

and, moreover, (see Proposition 10 in this appendix)

$$\overline{(V_p^\circ)^{-1}(0)} \subseteq \mathcal{C} \cup B_p. \quad (40)$$

We now wish to compute the largest invariant subset (in the sense of [36, Corollary 4.4]) of $V^{-1}(V^\infty) \cap \overline{(V_p^\circ)^{-1}(0)}$. For that purpose, consider a solution

$$\mathbb{R} \ni t \mapsto \mathbf{x}(t) \in V^{-1}(V^\infty) \cap \overline{(V_p^\circ)^{-1}(0)}. \quad (41)$$

Composing (34) with (41) yields a constant function $I \ni t \mapsto V(\mathbf{x}(t)) = V^\infty$, whose derivative is well defined, namely

$$I \ni t \mapsto \dot{V}(\mathbf{x}(t)) = v_{i,p}(\mathbf{x}(t))|_{\forall i \in H(\mathbf{x}(t))} = 0. \quad (42)$$

In fact, (42) implies that $v_{k,p}(\mathbf{x}) = v_{l,p}(\mathbf{x})$ for all $k, l \in H(\mathbf{x})$, which is not satisfied for any $\mathbf{x} \in B_p$ as defined in (39). This implies that \mathbf{x} in (41) does not belong to B_p . It then follows that the largest invariant subset of $V^{-1}(V^\infty) \cap \overline{(V_p^\circ)^{-1}(0)} \subseteq V^{-1}(V^\infty) \cap (\mathcal{C} \cup B_p)$ is, in fact, a subset of $V^{-1}(V^\infty) \cap \mathcal{C}$, which is independent of the mode $p \in \mathcal{P}$.

In brief, we used a Lyapunov function common to all modes, and verified that the largest invariant set for each mode is independent of the mode. Based on the previous observations, we can invoke [36, Corollary 4.7], from which it follows that (1) converges to $V^{-1}(V^\infty) \cap \mathcal{C} \subset \bar{\mathcal{B}}(r_0)^N \cap \mathcal{C}$. ■

Proposition 10 (Closure of the set where V_p° vanishes):

In order to verify that (40) holds, consider a convergent sequence in $(V_p^\circ)^{-1}(0)$, namely

$$\begin{aligned} \{\mathbf{x}^m \in (V_p^\circ)^{-1}(0)\}_{m \in \mathbb{N}}, \\ \mathbf{x}^\infty := \lim_{m \rightarrow \infty} \mathbf{x}^m \in A_p \cup B_p \cup \mathcal{C}. \end{aligned} \quad (43)$$

Let us prove (40) by assuming that $\mathbf{x}^\infty \in A_p$, which will lead to a contraction. For that purpose, denote $B_\epsilon(\mathbf{x}^\infty) := \{\mathbf{x} \in \mathbb{R}^{nN} : \|\mathbf{x} - \mathbf{x}^\infty\| < \epsilon\}$ as an open ball of size $\epsilon > 0$

around \mathbf{x}^∞ .

a) Since the sequence (43) belongs to $(V_p^\circ)^{-1}(0)$, it follows that for any $m \in \mathbb{N}$, there exist $i \in H(\mathbf{x}^m)$ s.t. $v_{i,p}(\mathbf{x}^m) = 0$. b) Since the sequence (43) is convergent, it follows that for any $\epsilon > 0$ there exists $M \in \mathbb{N}$ s.t., for all $m \geq M$, $\mathbf{x}^m \in B_\epsilon(\mathbf{x}^\infty)$. c) By definition of H in (4), it follows that for any $\mathbf{x} \in \mathbb{R}^{nN}$ there exists $\epsilon_1 > 0$ s.t., for all $\mathbf{y} \in B_{\epsilon_1}(\mathbf{x})$, $H(\mathbf{y}) \subseteq H(\mathbf{x})$. d) Since $v_{1,p}, \dots, v_{N,p}$ in (33) are continuous, it follows that for any $\mathbf{x} \in A_p$ (see (38)) there exists $\epsilon_2 \in (0, \epsilon_1)$ s.t., for all $\mathbf{y} \in B_{\epsilon_1}(\mathbf{x})$, $v_{i,p}(\mathbf{y}) < 0 \forall i \in H(\mathbf{y}) \subseteq H(\mathbf{x})$. e) Combining a) with d), it follows that for $\epsilon = \epsilon_2$ there exists $M \in \mathbb{N}$ s.t., for all $m \geq M$, $\mathbf{x}^m \in B_{\epsilon_2}(\mathbf{x}^\infty) \stackrel{d)}{\Rightarrow} v_{i,p}(\mathbf{x}^m) < 0 \forall i \in H(\mathbf{x}^m) \subseteq H(\mathbf{x}^\infty)$. However, e) contradicts a), which implies that $\mathbf{x}^\infty \notin A_p$.

APPENDIX II

PROOF OF PROPOSITION 1

A switching signal σ has an average dwell-time $\tau_D > 0$ and a chatter bound $N_0 \in \mathbb{N}$ if the number of switching times of σ in any open finite interval $(t_1, t_2) \subset \mathbb{R}$ is upper bounded by $\#_s(t_1, t_2) = N_0 + \frac{t_2 - t_1}{\tau_D}$ [32].

Proof: [of Proposition 1] For brevity, denote $\tau_D^{\min} = \min_{i \in \mathcal{N}} \tau_D^i$, $N_0^{\max} = \max_{i \in \mathcal{N}} N_0^i$, \mathcal{T} as the set of switching time instants of σ and \mathcal{T}^i as the set of switching time instants of $\mathcal{N}_i \circ \sigma$ for each $i \in \mathcal{N}$. If $N_0 = NN_0^{\max}$ and $\tau_D = \frac{1}{N} \tau_D^{\min}$, then $\#_s(t_1, t_2) = N \left(N_0^{\max} + \frac{t_2 - t_1}{\tau_D^{\min}} \right)$. Next, we show that this function indeed upper bounds the number of switches of σ . Consider then any open interval $(t_1, t_2) \subset \mathbb{R}_{\geq 0}$ where

$$0 < t_2 - t_1 \leq \frac{1}{N} \tau_D^{\min} \leq \tau_D^i, \forall i \in \mathcal{N}. \quad (44)$$

Consider a switch instant t in that interval, i.e. $t \in \mathcal{T}$ and $t \in (t_1, t_2)$. It follows that $t \in \mathcal{T}^j$ for some $j \in \mathcal{N}$. Under Assumption 2, the next N_0^j switches from \mathcal{T}^j may come only after a period $\tau_D^j \geq \tau_D^{\min} \geq \frac{1}{N} \tau_D^{\min}$; and, since (44) holds, it then follows that agent j can only switch N_0^j times in (t_1, t_2) . Since this is valid for any $j \in \mathcal{N}$, it follows that a maximum of NN_0^{\max} switches are possible in (t_1, t_2) . It also follows from (44) that $NN_0^{\max} \leq \#_s(t_1, t_2)$, and therefore $\#_s(t_1, t_2)$ upper bounds the number of switches in the interval (t_1, t_2) .

Consider now any open interval $(t_1, t_2) \subset \mathbb{R}_{\geq 0}$ where

$$\frac{k-1}{N} \tau_D^{\min} < t_2 - t_1 \leq \frac{k}{N} \tau_D^{\min}, \quad (45)$$

for some $k \in \{2, 3, \dots\}$. The interval (t_1, t_2) may be broken in k intervals of equal length, i.e., given $\Delta t = \frac{t_2 - t_1}{k}$, $(t_1, t_2) \setminus \cup_{l \in \{2, \dots, k\}} \{t_1 + (l-1)\Delta t\} = \cup_{l \in \{1, \dots, k\}} (t_1 + (l-1)\Delta t, t_1 + l\Delta t)$. For any $l \in \{1, \dots, k\}$, the interval $(t_1 + (l-1)\Delta t, t_1 + l\Delta t)$ has length $\Delta t \leq \frac{1}{N} \tau_D^{\min}$ where the inequality follows since (45) holds. Thus, we invoke the same reasoning as before to conclude that in each of those intervals only a maximum of NN_0^{\max} switches can occur, and, as such, only a total of kNN_0^{\max} switches can occur in (t_1, t_2) . It also follows from (45) that $kNN_0^{\max} \leq \#_s(t_1, t_2)$, and therefore $\#_s(t_1, t_2)$ upper bounds the number of switches in the interval (t_1, t_2) . ■

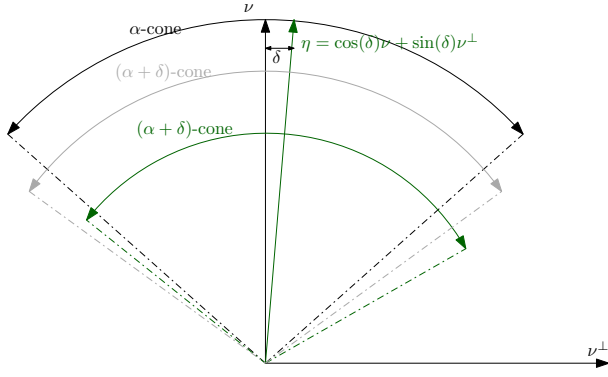


Fig. 7: Illustration of result in Proposition 11 for $n = 2$: closed $(\alpha + \delta)$ -cones formed by ν and η contain closed α -cone formed by ν ; also, ν and η are linearly independent.

APPENDIX III AUXILIARY RESULTS

Proposition 11: Let $\nu \in \mathcal{S}^n$ and $\alpha \in [0, \frac{\pi}{2})$. There exist $n + 1$ linearly independent unit vectors $\nu_1, \dots, \nu_{n+1} \in \mathcal{S}^n$ such that $\mathcal{C}(\alpha, \nu) \subset \mathcal{C}(\alpha + \delta, \nu_i) \subset \mathcal{C}(\frac{\pi}{2}, \nu_i)$ for all $i \in \{1, \dots, n + 1\}$ and for some $\delta \in (0, \frac{\pi}{2} - \alpha)$.

Proof: It is trivial to verify that $\mathcal{C}(\alpha, \nu) \subset \mathcal{C}(\alpha + \delta, \nu_i)$ for any $\delta \in (0, \frac{\pi}{2} - \alpha)$.

Recall that, by definition, $\langle \nu, \mathbf{n} \rangle > \cos(\alpha) > 0$ for all $\mathbf{n} \in \mathcal{C}(\alpha, \nu)$. Consider now the unit vector $\eta_i = \cos(\delta)\nu + \sin(\delta)\nu_i^\perp \in \mathcal{S}^n$, for some $\delta \in (0, \frac{\pi}{2} - \alpha)$ and where $\nu_i^\perp \in \mathcal{S}^n$ is a unit vector orthogonal to ν . Since $\delta \in (0, \frac{\pi}{2} - \alpha)$, it follows that $\cos(\delta) > 0$. Then $\langle \mathbf{n}, \eta_i \rangle = \cos(\delta)\langle \mathbf{n}, \nu \rangle + \sin(\delta)\langle \mathbf{n}, \nu_i^\perp \rangle > \cos(\delta)\cos(\alpha) - \sin(\delta)\sin(\alpha) = \cos(\alpha + \delta) > \cos(\alpha)$, for all $\mathbf{n} \in \mathcal{C}(\alpha, \nu)$. Since there are n unit vectors $\nu_1^\perp, \dots, \nu_n^\perp$ orthogonal to ν , it follows that we can find n linearly independent unit vectors η_1, \dots, η_n such that $\mathcal{C}(\alpha, \nu) \subset \mathcal{C}(\alpha + \delta, \eta_i)$ for all $i \in \{1, \dots, n\}$. Moreover, $\{\nu, \eta_1, \dots, \eta_n\}$ are $n + 1$ linearly independent unit vectors. ■

Figure 7 illustrates the result in Proposition 11 for $n = 2$. From Proposition 11 it follows that if a group of unit vectors (in \mathbb{R}^{n+1}) is contained in a closed α -cone for some $\alpha \in [0, \frac{\pi}{2})$, then we can find $n + 1$ larger (by δ) closed cones that contain the same group of unit vectors; i.e., given $\nu = (\nu_1, \dots, \nu_N) \in \bar{\mathcal{C}}(\alpha, \bar{\nu})^N$ for some $\bar{\nu} \in \mathbb{S}^n$, there exist $n + 1$ linearly independent unit vectors, $\{\bar{\nu}_1, \dots, \bar{\nu}_{n+1}\}$, such that $\nu \in \bar{\mathcal{C}}(\alpha + \delta, \bar{\nu}_i)^N$ for all $i \in \{1, \dots, n + 1\}$ and for some $\delta \in (0, \frac{\pi}{2} - \alpha)$.

Proof: [of Proposition 7] *Sufficiency:* Regarding (a), if $\nu_2 = \nu_1$, then $\langle \nu, \Pi(\nu_1)\nu_2 \rangle = \langle \bar{\nu}, \Pi(\nu_1)\nu_1 \rangle = 0$. Regarding (b), if $\nu_1 \neq \nu_2$ then $\langle \nu_1, \nu_2 \rangle < 1$. Consider then the following two cases: (1) $\langle \nu, \nu_1 \rangle = \langle \nu, \nu_2 \rangle$ and (2) $\langle \nu, \nu_1 \rangle \neq \langle \nu, \nu_2 \rangle$. For case (1), it follows that $\langle \nu, \Pi(\nu_1)\nu_2 \rangle = \langle \nu, \nu_2 \rangle - \langle \nu, \nu_1 \rangle \nu_1^T \nu_2 = \langle \nu, \nu_1 \rangle (1 - \nu_1^T \nu_2) > 0$, where the inequality applies since $\nu_1^T \nu_2 < 1$ and since, by assumption $0 < \langle \nu, \nu_1 \rangle$. For case (2), the Proposition's assumption becomes $0 < \langle \nu, \nu_1 \rangle < \langle \nu, \nu_2 \rangle$. Then, since $\langle \nu_1, \nu_2 \rangle < 1$, it follows that $0 < \langle \nu, \nu_1 \rangle < \langle \nu, \nu_1 \rangle \langle \nu_1, \nu_2 \rangle \Rightarrow \langle \nu, \Pi(\nu_1)\nu_2 \rangle > 0$. *Necessity:* Regarding

(a), if we assume on the contrary that $\nu_1 \neq \nu_2$, then it follows from before that $\langle \nu, \Pi(\nu_1)\nu_2 \rangle = \langle \nu, \nu_2 \rangle - \langle \nu, \nu_1 \rangle \nu_1^T \nu_2 > 0$, which implies that $\nu_1 = \nu_2$. Similarly, regarding (b), if we assume on the contrary that $\nu_1 = \nu_2$, then it follows from before that $\langle \nu, \Pi(\nu_1)\nu_2 \rangle = \langle \bar{\nu}, \Pi(\nu_1)\nu_1 \rangle = 0$, which implies that $\nu_1 \neq \nu_2$. ■

APPENDIX IV QUATERNIONS

This section provides some auxiliary results that are useful in Section IV-A. Recall the map (16). Given $\mathcal{R} \in \tilde{\mathbb{S}}\mathbb{O}(3)$, consider then $\mathbf{q} = T_r^q(\mathcal{R})$, and for $i \in \{1, 2, 3, 4\}$, denote the i^{th} component of the quaternion as $\mathbf{q}_i = T_r^{qi}(\mathcal{R})$ (denote dT_r^{qi} as the derivative of the function T_r^{qi}). Its kinematics are given by

$$\begin{aligned} \dot{\mathbf{q}}_i(\mathbf{q}, \omega) &\equiv \mathbf{f}_{qi}(\mathbf{q}, \omega) \\ &:= \text{tr}(dT_r^{qi}(\mathcal{R})\mathbf{f}_{\mathcal{R}}(\mathcal{R}, \omega)^T) |_{\mathcal{R}=T_r^q(\mathbf{q})} \\ &\stackrel{(10)}{=} -\text{tr}(dT_r^{qi}(\mathcal{R})\mathcal{S}(\omega)\mathcal{R}^T) |_{\mathcal{R}=T_r^q(\mathbf{q})} \\ &= -\text{tr}(\mathcal{R}^T dT_r^{qi}(\mathcal{R})\mathcal{S}(\omega)) |_{\mathcal{R}=T_r^q(\mathbf{q})} \\ &= \langle \mathcal{S}^{-1}(X - X^T), \omega \rangle |_{X=\mathcal{R}^T dT_r^{qi}(\mathcal{R})} |_{\mathcal{R}=T_r^q(\mathbf{q})} \end{aligned}$$

where we have made use of the facts that $\text{tr}(AB) = \text{tr}(BA)$ and $\text{tr}(A\mathcal{S}(\mathbf{a})) = \langle \mathbf{a}, \mathcal{S}^{-1}(A - A^T) \rangle$ for any $A, B \in \mathbb{R}^{3 \times 3}$ and $\mathbf{a} \in \mathbb{R}^3$. Denote $\tilde{\mathbb{S}}^3$ as the image of the map T_r^{qi} . Collectively, the kinematics of the quaternion

$$\dot{\mathbf{q}} \equiv \mathbf{f}_q : \tilde{\mathbb{S}}^3 \times \mathbb{R}^3 \ni (\mathbf{q}, \omega) \mapsto \mathbf{f}_q(\mathbf{q}, \omega) \in T_q \tilde{\mathbb{S}}^3$$

are given by

$$\begin{aligned} \dot{\mathbf{q}}(\mathbf{q}, \omega) &\equiv \mathbf{f}_q(\mathbf{q}, \omega) := (\mathbf{f}_{q1}(\mathbf{q}, \omega), \dots, \mathbf{f}_{q4}(\mathbf{q}, \omega)) \\ &= \frac{1}{2} Q(\mathbf{q}) [\mathbf{0}_{3 \times 1} \mathbf{I}_3]^T \omega \end{aligned}$$

with Q as defined in (18), and which can be extended from $\tilde{\mathbb{S}}^3$ to \mathbb{S}^3 .

It can be verified that, given any $\mathbf{q}_1, \mathbf{q}_2 \in \mathbb{S}^3$, and $\mathcal{R}_1 = T_q^r(\mathbf{q}_1)$ and $\mathcal{R}_2 = T_q^r(\mathbf{q}_2)$, it follows that

$$\mathcal{R}_1^T \mathcal{R}_2 = T_q^r(Q(\mathbf{q}_1)^T \mathbf{q}_2). \quad (46)$$

It can also be easily verified that, given $\mathbf{q} \in \mathbb{S}^3$, and $\mathcal{R} = T_q^r(\mathbf{q})$, it follows that

$$\begin{aligned} \theta(\mathbf{I}_3, \mathcal{R}) |_{\mathcal{R}=T_q^r(\mathbf{q})} &= \arccos(2 \langle \mathbf{q}, \mathbf{e}_1 \rangle^2 - 1), \\ \mathcal{S}^{-1}(\mathcal{R} - \mathcal{R}^T) |_{\mathcal{R}=T_q^r(\mathbf{q})} &= 2 \langle \mathbf{q}, \mathbf{e}_1 \rangle [\mathbf{0}_{3 \times 1} \mathbf{I}] Q(\mathbf{q}) \mathbf{e}_1. \end{aligned}$$

Combining the latter with (46), it follows that

$$\begin{aligned} \theta(\mathbf{I}_3, \mathcal{R}_1^T \mathcal{R}_2) |_{\mathcal{R}_1^T \mathcal{R}_2 = T_q^r(Q(\mathbf{q}_1)^T \mathbf{q}_2)} &= \\ &= \arccos(2 \langle \mathbf{e}_1, Q(\mathbf{q}_1)^T \mathbf{q}_2 \rangle^2 - 1) \\ &= \arccos(2 \langle \mathbf{q}_1, \mathbf{q}_2 \rangle^2 - 1), \end{aligned} \quad (47)$$

and that (note, from (18), that $Q(\mathbf{q})^T \mathbf{e}_1 = \mathbf{q}$)

$$\begin{aligned} \mathcal{S}^{-1}(\mathcal{R}_1^T \mathcal{R}_2 - (\mathcal{R}_1^T \mathcal{R}_2)^T) |_{\mathcal{R}_1^T \mathcal{R}_2 = T_q^r(Q(\mathbf{q}_1)^T \mathbf{q}_2)} &= \\ &= 2 \langle \mathbf{e}_1, Q(\mathbf{q}_1)^T \mathbf{q}_2 \rangle [\mathbf{0}_{3 \times 1} \mathbf{I}] Q(Q(\mathbf{q}_1)^T \mathbf{q}_2)^T \mathbf{e}_1 \\ &= 2 \langle \mathbf{q}_1, \mathbf{q}_2 \rangle [\mathbf{0}_{3 \times 1} \mathbf{I}] Q(\mathbf{q}_1)^T \mathbf{q}_2. \end{aligned} \quad (48)$$

APPENDIX V

CONSENSUS IN \mathbb{R}^n CASTED AS SYNCHRONIZATION IN \mathbb{S}^n

In this section, we consider a group of N agents operating in \mathbb{R}^n , for some $n \in \mathbb{N}$. For each $i \in \mathcal{N}$, $\mathbf{u}_i : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}^n$ is the body-framed linear velocity of agent i , which can be actuated. Consider then $\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_N) : \mathbb{R}_{\geq 0} \mapsto (\mathbb{R}^n)^N$, where each position $\mathbf{x}_i : \mathbb{R}_{\geq 0} \ni t \mapsto \mathbf{x}_i(t) \in \mathbb{R}^n$ evolves according to

$$\dot{\mathbf{x}}_i(t) = \mathcal{R}_i \mathbf{u}_i(t) =: \mathbf{f}_i(\mathbf{u}_i(t)), \quad (49)$$

where $\mathbf{f}_i : \mathbb{R}^n \ni \mathbf{u} \mapsto \mathbf{f}_i(\mathbf{u}) := \mathcal{R}_i \mathbf{u} \in \mathbb{R}^n$ and where $\mathcal{R}_i \in \mathbb{SO}(3)$ represents the body orientation frame of agent i w.r.t. some unknown inertial orientation frame. In physical terms, $\mathcal{R}_i \mathbf{u}_i$ corresponds to the inertial-framed linear velocity of agent i , and we assume that agent i is unaware of its orientation w.r.t. the inertial orientation frame.

If, at a time instant $t \in \mathbb{R}_{\geq 0}$, agent $i \in \mathcal{N}$ is aware of the relative position between itself and another agent j , then $j \in \mathcal{N}_i(\sigma(t))$, where $\sigma(t)$ encodes the network graph at time t . At each time instant $t \in \mathbb{R}_{\geq 0}$ and for $\mathbf{x} \in (\mathbb{R}^n)^N$, each agent $i \in \mathcal{N}$ measures $\mathbf{h}_i(t, \mathbf{x})$, where $\mathbf{h}_i(t, \cdot) : (\mathbb{R}^n)^N \ni \mathbf{x} \mapsto \mathbf{h}_i(t, \mathbf{x}) \in (\mathbb{R}^n)^{|\mathcal{N}_i(\sigma(t))|}$ is defined as

$$\mathbf{h}_i(t, \mathbf{x}) := (\mathbf{h}_{ij_1}(\mathbf{x}), \dots, \mathbf{h}_{ij_{|\mathcal{N}_i(\sigma(t))|}}(\mathbf{x})) \in (\mathbb{R}^n)^{|\mathcal{N}_i(\sigma(t))|}, \quad (50)$$

where $\{j_1, \dots, j_{|\mathcal{N}_i(\sigma(t))|}\} \equiv \mathcal{N}_i(\sigma(t))$ and where $\mathbf{h}_{ij} : (\mathbb{R}^n)^N \ni \mathbf{x} := (\mathbf{x}_1, \dots, \mathbf{x}_N) \mapsto \mathbf{h}_{ij}(\mathbf{x}) \in \mathbb{R}^n$ is defined as

$$\mathbf{h}_{ij}(\mathbf{x}) := \mathcal{R}_i^T (\mathbf{x}_j - \mathbf{x}_i) \in \mathbb{R}^n, \quad (51)$$

for each $j \in \mathcal{N}_i(\sigma(t))$. Thus, at each time instant $t \in \mathbb{R}_{\geq 0}$, an agent $i \in \mathcal{N}$ makes $|\mathcal{N}_i(\sigma(t))|$ measurements, and each measurement corresponds to a distance vector between agent i and one of its neighbors, projected onto agent's i orientation frame. Notice that (51) does not require a common reference frame among agents, i.e., agents do not need to agree on a common origin and orientation frame.

Problem 3: For each time instant $t \in \mathbb{R}_{\geq 0}$ and for each $i \in \mathcal{N}$, design time-varying decentralized feedback laws $\mathbf{u}_i^h(t, \cdot) : (\mathbb{R}^n)^{|\mathcal{N}_i(\sigma(t))|} \mapsto \mathbb{R}^n$, such that asymptotic consensus of $\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_N) : \mathbb{R}_{\geq 0} \mapsto (\mathbb{R}^n)^N$ is accomplished, where $\dot{\mathbf{x}}_i(t) = \mathbf{f}_i(\mathbf{u}_i^h(\mathbf{h}_i(t, \mathbf{x}(t))))$ for every $i \in \mathcal{N}$ and with $\mathbf{h}_i(t, \cdot)$ as defined in (50).

For each $t \in \mathbb{R}_{\geq 0}$ and each agent $i \in \mathcal{N}$, consider the control law $\mathbf{u}_i^h(t, \cdot) : \mathbb{R}^{n|\mathcal{N}_i(\sigma(t))|} \ni \mathbf{h}_i := (\mathbf{h}_{ij_1}, \dots, \mathbf{h}_{ij_{|\mathcal{N}_i(\sigma(t))|}}) \mapsto \mathbf{u}_i^h(t, \mathbf{h}_i) \in \mathbb{R}^n$ defined as

$$\mathbf{u}_i^h(t, \mathbf{h}_i) := - \sum_{j \in \mathcal{N}_i(\sigma(t))} w_{ij}(\|\mathbf{h}_{ij}\|) \mathbf{h}_{ij}, \quad (52)$$

where $w_{ij} : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}_{\geq 0}$ is a weight function agent i assigns to the position error between itself and its neighbor j . This weight may be used, for example, to bound the actuation: indeed, if $\mathbb{R}_{\geq 0} \ni x \mapsto w_{ij}(x) := \frac{\sigma}{\sqrt{\sigma^2 + x^2}}$ for all $j \in \mathcal{N}$ and some $\sigma > 0$, then $\|\mathbf{u}_i^h(\cdot, \cdot)\| \leq N\sigma$ (since $\sup_{x \geq 0} w_{ij}(x)x = \sigma$). Denote \mathbf{u}_i^{cl} as the composition of the control law (52) with the measurement function (50), i.e., $\mathbf{u}_i^{cl} : \mathbb{R}_{\geq 0} \times (\mathbb{R}^n)^N \ni$

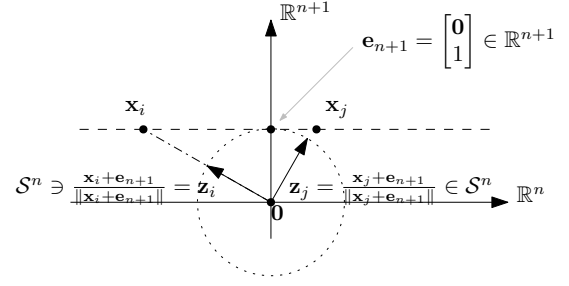


Fig. 8: Casting consensus in \mathbb{R}^n as synchronization in \mathbb{S}^n .

$(t, \mathbf{x}) \mapsto \mathbf{u}_i^{cl}(t, \mathbf{x}) := \mathbf{u}_i^h(t, \mathbf{h}_i(t, \mathbf{x})) \in \mathbb{R}^n$. It follows that

$$\mathbf{u}_i^{cl}(t, \mathbf{x}) = - \mathcal{R}_i^T \sum_{j \in \mathcal{N}_i(\sigma(t))} w_{ij}(\|\mathbf{x}_i - \mathbf{x}_j\|) (\mathbf{x}_i - \mathbf{x}_j) \quad (53)$$

By composing (49) with (53), it follows that

$$\dot{\mathbf{x}}_i(t) = \sum_{j \in \mathcal{N}_i(\sigma(t))} w_{ij}(\|\mathbf{x}_j(t) - \mathbf{x}_i(t)\|) (\mathbf{x}_j(t) - \mathbf{x}_i(t)) \quad (54)$$

which is not in the form (8). In order to write the closed loop dynamics (54) as in (8), we perform a transformation which is discussed next.

In order to analyze consensus in \mathbb{R}^n under the same framework as synchronization in \mathbb{S}^2 and $\mathbb{SO}(3)$, we now perform a change of variables that serves only the purpose of analysis. Consider then the unit vector $\mathbf{e}_{n+1} = (\mathbf{0}_n, 1) \in \mathbb{S}^n \in \mathbb{R}^{n+1}$ and the matrix $P = [\mathbf{I}_n \ \mathbf{0}_n]^T \in \mathbb{R}^{(n+1) \times n}$. Consider also the mapping $\mathbf{h} : \mathbb{R}^n \ni \mathbf{x} \mapsto \mathbf{h}(\mathbf{x}) \in \mathcal{C}(\frac{\pi}{2}, \mathbf{e}_{n+1}) \subset \mathbb{S}^n \subset \mathbb{R}^{n+1}$, defined as

$$\mathbf{h}(\mathbf{x}) := \frac{P\mathbf{x} + \mathbf{e}_{n+1}}{\|P\mathbf{x} + \mathbf{e}_{n+1}\|}, \quad (55)$$

where $\|P\mathbf{x} + \mathbf{e}_{n+1}\| = \sqrt{1 + \|\mathbf{x}\|^2} \geq 1 > 0$ for all $\mathbf{x} \in \mathbb{R}^n$. This transformation is illustrated in Fig.8. Notice that \mathbf{h} is, in fact, a diffeomorphism between \mathbb{R}^n and $\mathcal{C}(\frac{\pi}{2}, \mathbf{e}_{n+1})$, with $\mathbf{h}^{-1} : \mathcal{C}(\frac{\pi}{2}, \mathbf{e}_{n+1}) \ni \mathbf{z} \mapsto \mathbf{h}^{-1}(\mathbf{z}) \in \mathbb{R}^n$ given by

$$\mathbf{h}^{-1}(\mathbf{z}) = P^T \left(\frac{1}{\langle \mathbf{z}, \mathbf{e}_{n+1} \rangle} \mathbf{z} - \mathbf{e}_{n+1} \right). \quad (56)$$

It follows from (55) and (56) that, for any $\mathbf{z}_i, \mathbf{z}_j \in \mathcal{C}(\frac{\pi}{2}, \mathbf{e}_{n+1})$,

$$\begin{aligned} & \frac{P\mathbf{x}_j - P\mathbf{x}_i}{\|P\mathbf{x}_i + \mathbf{e}_{n+1}\|} \Big|_{\mathbf{x}_i = \mathbf{h}^{-1}(\mathbf{z}_i), \mathbf{x}_j = \mathbf{h}^{-1}(\mathbf{z}_j)} \\ &= \left(\frac{\|P\mathbf{x}_j + \mathbf{e}_{n+1}\|}{\|P\mathbf{x}_i + \mathbf{e}_{n+1}\|} \frac{P\mathbf{x}_j}{\|P\mathbf{x}_j + \mathbf{e}_{n+1}\|} - \frac{P\mathbf{x}_i}{\|P\mathbf{x}_i + \mathbf{e}_{n+1}\|} \right) \Big|_{\mathbf{x}_i = \mathbf{h}^{-1}(\mathbf{z}_i), \mathbf{x}_j = \mathbf{h}^{-1}(\mathbf{z}_j)} \\ &= \mathbf{z}_j \frac{\langle \mathbf{z}_i, \mathbf{e}_{n+1} \rangle}{\langle \mathbf{z}_j, \mathbf{e}_{n+1} \rangle} - \mathbf{z}_i. \end{aligned} \quad (57)$$

Let $\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_N) \in (\mathbb{R}^n)^N$ and $\mathbf{z} = (\mathbf{z}_1, \dots, \mathbf{z}_N) \in \mathcal{C}(\frac{\pi}{2}, \mathbf{e}_{n+1})^N$ where $\mathbf{z} = \mathbf{h}^N(\mathbf{x}) \Leftrightarrow \mathbf{x} = (\mathbf{h}^{-1})^N(\mathbf{z})$. It holds that, for any $i \in \mathcal{N}$, (note that $d\mathbf{h}(\mathbf{x}_i) =$

$$\begin{aligned}
& \frac{1}{\|P\mathbf{x}_i + \mathbf{e}_{n+1}\|} \Pi(\mathbf{h}(\mathbf{x}_i)) P \\
& d\mathbf{h}(\mathbf{x}_i) \mathbf{f}_i(\mathbf{u}_i^{cl}(t, \mathbf{x}))|_{\mathbf{x}=(\mathbf{h}^{-1})^N(\mathbf{z})} = \\
(49) &= \Pi(\mathbf{h}(\mathbf{x}_i)) \frac{P\mathcal{R}_i \mathbf{u}_i^{cl}(t, \mathbf{x})}{\|P\mathbf{x}_i + \mathbf{e}_{n+1}\|} |_{\mathbf{x}=(\mathbf{h}^{-1})^N(\mathbf{z})} \\
(54) &= \sum_{j \in \mathcal{N}_i(\sigma(t))} \Pi(\mathbf{h}(\mathbf{x}_i)) w_{ij} (\|P\mathbf{x}_j - P\mathbf{x}_i\|) \frac{P\mathbf{x}_j - P\mathbf{x}_i}{\|P\mathbf{x}_i + \mathbf{e}_{n+1}\|} |_{\mathbf{x}=(\mathbf{h}^{-1})^N(\mathbf{z})} \\
(57) &= \sum_{j \in \mathcal{N}_i(\sigma(t))} \Pi(\mathbf{z}_i) w_{ij} \left(\left\| \frac{\mathbf{z}_j}{\langle \mathbf{e}_{n+1}, \mathbf{z}_j \rangle} - \frac{\mathbf{z}_i}{\langle \mathbf{e}_{n+1}, \mathbf{z}_i \rangle} \right\| \right) \left(\frac{\langle \mathbf{e}_{n+1}, \mathbf{z}_i \rangle}{\langle \mathbf{e}_{n+1}, \mathbf{z}_j \rangle} \mathbf{z}_j - \mathbf{z}_i \right) \\
&= \sum_{j \in \mathcal{N}_i(\sigma(t))} \frac{\langle \mathbf{e}_{n+1}, \mathbf{z}_i \rangle}{\langle \mathbf{e}_{n+1}, \mathbf{z}_j \rangle} w_{ij} \left(\left\| \frac{\mathbf{z}_j}{\langle \mathbf{e}_{n+1}, \mathbf{z}_j \rangle} - \frac{\mathbf{z}_i}{\langle \mathbf{e}_{n+1}, \mathbf{z}_i \rangle} \right\| \right) \Pi(\mathbf{z}_i) \mathbf{z}_j \\
&=: \sum_{j \in \mathcal{N}_i(\sigma(t))} \tilde{w}_{ij}(\mathbf{z}_i, \mathbf{z}_j) \Pi(\mathbf{z}_i) \mathbf{z}_j \\
(8) &=: \tilde{\mathbf{f}}_{i, \sigma(t)}(\mathbf{z})
\end{aligned}$$

where, in the one to last step, we defined $\tilde{w}_{ij}(\mathbf{z}_i, \mathbf{z}_j) := \frac{\langle \mathbf{e}_{n+1}, \mathbf{z}_i \rangle}{\langle \mathbf{e}_{n+1}, \mathbf{z}_j \rangle} w_{ij} \left(\left\| \frac{\mathbf{z}_j}{\langle \mathbf{e}_{n+1}, \mathbf{z}_j \rangle} - \frac{\mathbf{z}_i}{\langle \mathbf{e}_{n+1}, \mathbf{z}_i \rangle} \right\| \right)$, which satisfies (9) for $\alpha = \frac{\pi}{2}$ and for $\bar{\nu} = \mathbf{e}_{n+1} \in \mathbb{S}^n$. We have thus casted this problem in the form (8) with $\nu \equiv \mathbf{z} \in (\mathbb{S}^n)^N$.

Remark 12: Unlike synchronization in \mathbb{S}^2 and $\mathbb{SO}(3)$, the unit vectors in this section are by construction contained in a $\frac{\pi}{2}$ -cone formed by the unit vector \mathbf{e}_{n+1} (see the co-domain of \mathbf{h} in (55)). Also, note that \mathbf{h} in (55) may be defined with other vectors other than \mathbf{e}_{n+1} , i.e., $\mathbb{R}^n \ni \mathbf{x} \mapsto \mathbf{h}(\mathbf{x}) = \frac{P\mathbf{x} + \tilde{\mathbf{e}}}{\|P\mathbf{x} + \tilde{\mathbf{e}}\|}$ with some $\tilde{\mathbf{e}} \in \mathbb{S}^n$ satisfying $\langle \mathbf{e}_{n+1}, \tilde{\mathbf{e}} \rangle \neq 0$ also works as an alternative transformation.

REFERENCES

- [1] J. R. Lawton and R. W. Beard, "Synchronized multiple spacecraft rotations," *Automatica*, vol. 38, no. 8, pp. 1359–1364, 2002.
- [2] A. Abdessameud and A. Tayebi, "Attitude synchronization of a group of spacecraft without velocity measurements," *IEEE Transactions on Automatic Control*, vol. 54, no. 11, pp. 2642–2648, 2009.
- [3] N. Leonard, D. Paley, F. Lekien, R. Sepulchre, D. Fratantoni, and R. Davis, "Collective motion, sensor networks, and ocean sampling," *Proceedings of the IEEE*, vol. 95, no. 1, pp. 48–74, Jan 2007.
- [4] T. Hatanaka, Y. Igarashi, M. Fujita, and M. Spong, "Passivity-based pose synchronization in three dimensions," *IEEE Transactions on Automatic Control*, vol. 57, no. 2, pp. 360–375, 2012.
- [5] W. Ren, "Distributed attitude consensus among multiple networked spacecraft," in *American Control Conference*. IEEE, June 2006, pp. 1760–1765.
- [6] A. Sarlette, R. Sepulchre, and N. E. Leonard, "Autonomous rigid body attitude synchronization," *Automatica*, vol. 45, no. 2, pp. 572–577, 2009.
- [7] D. V. Dimarogonas, P. Tsiotras, and K. Kyriakopoulos, "Leader-follower cooperative attitude control of multiple rigid bodies," *Systems & Control Letters*, vol. 58, no. 6, pp. 429–435, 2009.
- [8] H. Bai, M. Arcak, and J. T. Wen, "Rigid body attitude coordination without inertial frame information," *Automatica*, vol. 44, no. 12, pp. 3170–3175, 2008.
- [9] R. Tron, B. Afsari, and R. Vidal, "Intrinsic consensus on $\mathbb{SO}(3)$ with almost-global convergence," in *Conference on Decision and Control*, 2012, pp. 2052–2058.
- [10] S. Chung, S. Bandyopadhyay, I. Chang, and F. Hadaegh, "Phase synchronization control of complex networks of lagrangian systems on adaptive digraphs," *Automatica*, vol. 49, no. 5, pp. 1148–1161, 2013.
- [11] A. K. Bondhus, K. Y. Pettersen, and J. T. Gravdahl, "Leader/follower synchronization of satellite attitude without angular velocity measurements," in *Conference on Decision and Control and European Control Conference*. IEEE, 2005, pp. 7270–7277.
- [12] T. Krogstad and J. Gravdahl, "Coordinated attitude control of satellites in formation," in *Group Coordination and Cooperative Control*. Springer, 2006, pp. 153–170.
- [13] K. Oh and H. Ahn, "Formation control and network localization via orientation alignment," *IEEE Transactions on Automatic Control*, vol. 59, no. 2, pp. 540–545, 2014.
- [14] R. Olfati-Saber, "Swarms on sphere: A programmable swarm with synchronous behaviors like oscillator networks," in *Conference on Decision and Control*. IEEE, 2006, pp. 5060–5066.
- [15] N. Moshtagh and A. Jadbabaie, "Distributed geodesic control laws for flocking of nonholonomic agents," *Transactions on Automatic Control*, vol. 52, no. 4, pp. 681–686, 2007.
- [16] D. A. Paley, "Stabilization of collective motion on a sphere," *Automatica*, vol. 45, no. 1, pp. 212–216, 2009.
- [17] W. Li and M. W. Spong, "Unified cooperative control of multiple agents on a sphere for different spherical patterns," *Transactions on Automatic Control*, vol. 59, no. 5, pp. 1283–1289, 2014.
- [18] A. Sarlette, S. E. Tuna, V. Blondel, and R. Sepulchre, "Global synchronization on the circle," in *Proceedings of the 17th IFAC world congress*, 2008, pp. 9045–9050.
- [19] A. Sarlette and R. Sepulchre, "Synchronization on the circle," *arXiv preprint arXiv:0901.2408*, 2009.
- [20] F. Dörfler and F. Bullo, "Synchronization in complex networks of phase oscillators: A survey," *Automatica*, vol. 50, no. 6, pp. 1539–1564, 2014.
- [21] L. Moreau, "Stability of continuous-time distributed consensus algorithms," in *Conference on Decision and Control*, vol. 4. IEEE, 2004, pp. 3998–4003.
- [22] —, "Stability of multiagent systems with time-dependent communication links," *Transactions on Automatic Control*, vol. 50, no. 2, pp. 169–182, 2005.
- [23] H. Zhang, C. Zhai, and Z. Chen, "A general alignment repulsion algorithm for flocking of multi-agent systems," *IEEE Transactions on Automatic Control*, vol. 56, no. 2, pp. 430–435, 2011.
- [24] P. O. Pereira and D. V. Dimarogonas, "Family of controllers for attitude synchronization on the sphere," *Automatica*, vol. 75, pp. 271 – 281, 2017. [Online]. Available: <http://www.sciencedirect.com/science/article/pii/S0005109816303788>
- [25] J. Thunberg, W. Song, E. Montijano, Y. Hong, and X. Hu, "Distributed attitude synchronization control of multi-agent systems with switching topologies," *Automatica*, vol. 50, no. 3, pp. 832–840, 2014.
- [26] Y. Igarashi, T. Hatanaka, M. Fujita, and M. W. Spong, "Passivity-based attitude synchronization in $\mathbb{SE}(3)$," *IEEE Transactions on Control Systems Technology*, vol. 17, no. 5, pp. 1119–1134, 2009.
- [27] R. Sepulchre, "Consensus on nonlinear spaces," *Annual reviews in control*, vol. 35, no. 1, pp. 56–64, 2011.
- [28] A. Sarlette and R. Sepulchre, "Consensus optimization on manifolds," *SIAM Journal on Control and Optimization*, vol. 48, no. 1, pp. 56–76, 2009.
- [29] Z. Lin, B. Francis, and M. Maggiore, "State agreement for continuous-time coupled nonlinear systems," *SIAM Journal on Control and Optimization*, vol. 46, no. 1, pp. 288–307, 2007.
- [30] J. P. Hespanha, "Uniform stability of switched linear systems: Extensions of lasalle's invariance principle," *IEEE Transactions on Automatic Control*, vol. 49, no. 4, pp. 470–482, 2004.
- [31] A. Bacciotti and L. Mazzi, "An invariance principle for nonlinear switched systems," *Systems & Control Letters*, vol. 54, no. 11, pp. 1109–1119, 2005.
- [32] J. L. Mancilla-Aguilar and R. A. García, "An extension of lasalle's invariance principle for switched systems," *Systems & Control Letters*, vol. 55, no. 5, pp. 376–384, 2006.
- [33] N. Fischer, R. Kamalapurkar, and W. E. Dixon, "Lasalle-yoshizawa corollaries for nonsmooth systems," *IEEE Transactions on Automatic Control*, vol. 9, no. 58, pp. 2333–2338, 2013.
- [34] M. Field and D. Pence, "Spacecraft attitude, rotations and quaternions," *UMAP Modules*, vol. 5, no. 2, p. 130, 1984.
- [35] P. O. Pereira, D. Boskos, and D. V. Dimarogonas, "A common framework for attitude synchronization of unit vectors in networks with switching topology," in *2016 IEEE 55th Conference on Decision and Control (CDC)*, Dec 2016, pp. 3530–3536. [Online]. Available: <http://ieeexplore.ieee.org/document/7798799/>
- [36] R. Goebel, R. G. Sanfelice, and A. R. Teel, "Invariance principles for switching systems via hybrid systems techniques," *Systems & Control Letters*, vol. 57, no. 12, pp. 980–986, 2008.
- [37] R. G. Sanfelice, R. Goebel, and A. R. Teel, "Invariance principles for hybrid systems with connections to detectability and asymptotic stability," *IEEE Transactions on Automatic Control*, vol. 52, no. 12, pp. 2282–2297, 2007.